Option Pricing

Simple Arbitrage Relations

Payoffs to Call and Put Options

Black-Scholes Model

Put-Call Parity

Implied Volatility

Options: Definitions

A call option gives the buyer the right, but not the obligation,

- To purchase a specific asset (e.g., 100 shares of Facebook (FB) stock, whose current price, $S = \$184.92$)
- For a prespecified price (exercise or "strike" price, e.g. $X = \$190$ per share)
- On a specific future date (the maturity date, e.g., T = Jan 19, 2019)

American vs. European options

- American options can be exercised anytime up to maturity; European options can only be exercised on the maturity date

Arbitrage Restrictions on Call Prices

1) $C \geq 0$

Consider the portfolio formed by buying the call option

Today	Later: $S^* \leq K$	Later: $S^* > K$
$-C$	0	$S^* - K > 0$

Arbitrage Restrictions on Call Prices

2) $C \leq S$

Consider the following portfolio:
Buy the stock and sell the call

Today	Later: $S^* \leq K$	Later: $S^* > K$
$C - S$	S^*	$S^* - (S^* - K) = K$

Arbitrage Restrictions on Call Prices

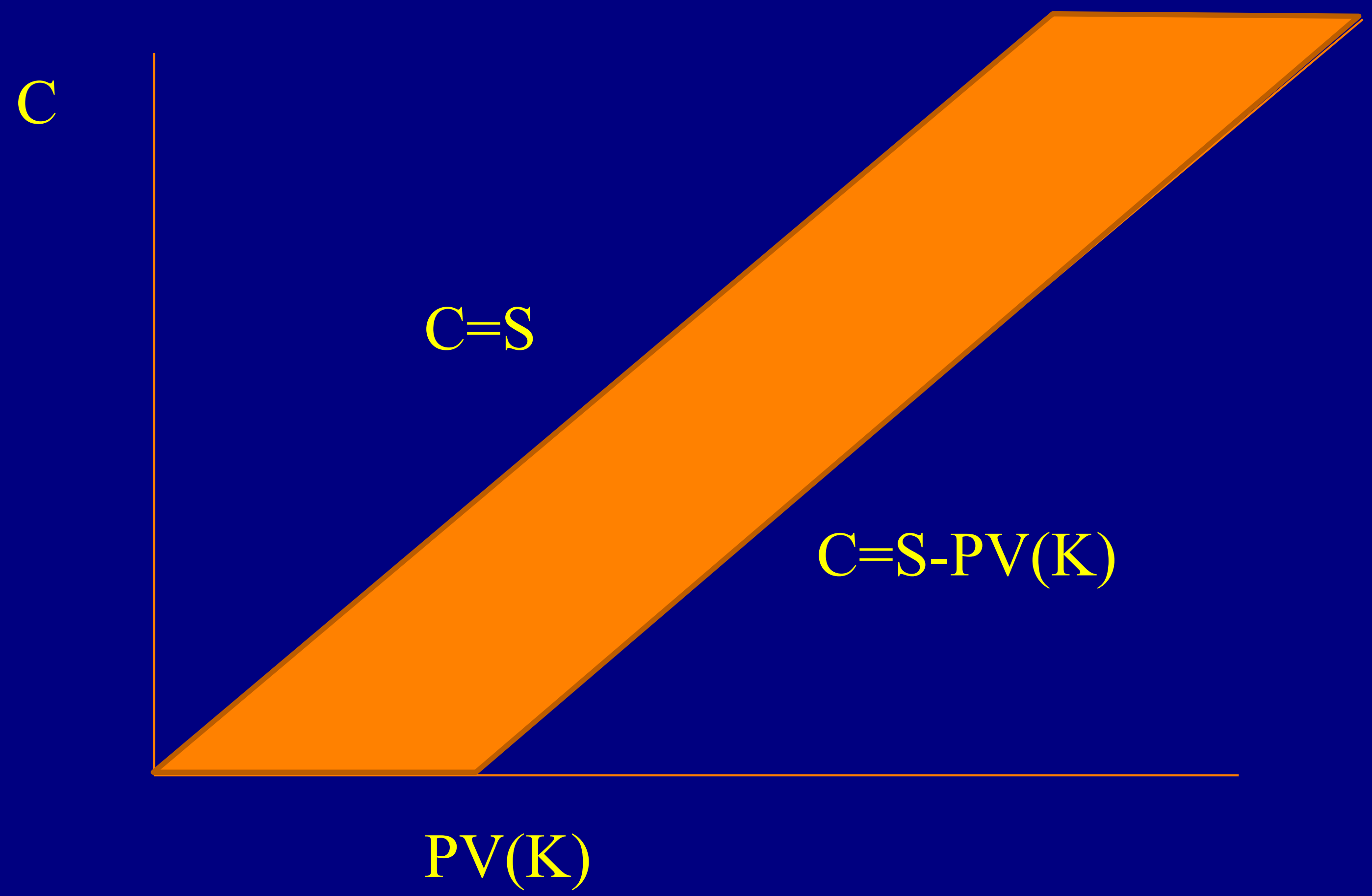
3) $C \geq S - PV(K)$

Consider the following portfolio:
Buy the call, sell the stock, then lend PV(K)

Today	Later	
	$S^* \leq K$	$S^* > K$
$S - C - PV(K)$	$K - S^*$	$(S^* - K) - S^* + K$
	> 0	$= 0$

Arbitrage Restrictions on Call Prices

1) $C \geq 0$
2) $C \leq S$
3) $C \geq S - PV(K)$



Early Exercise of American Options

Suppose you own a call option and you want to close out your position

- You can exercise and receive S – K
- Or, you can sell your option for its current market price C
- You choose the alternative that yields the greatest profit
  – Exercise if C < S – K
  – Sell if C > S - K

Arbitrage Restrictions on American Call Prices

Suppose $C < S - K$ between ex-dividend days

Then buy 1 call, short the stock, and lend K

Close out the position just before the ex-dividend day

<u>Today</u>	<u>at Ex-dividend Day</u>	
	<u>$S^* \leq K$</u>	<u>$S^* > K$</u>
$-C + S - K \geq 0$	$-S^*+(1+r)K > 0$	$(S^*-K)-S^*+(1+r)K = rK > 0$

Arbitrage Restrictions on American Call Prices

$C > S - K$ except at expiration or just prior to an ex-dividend day

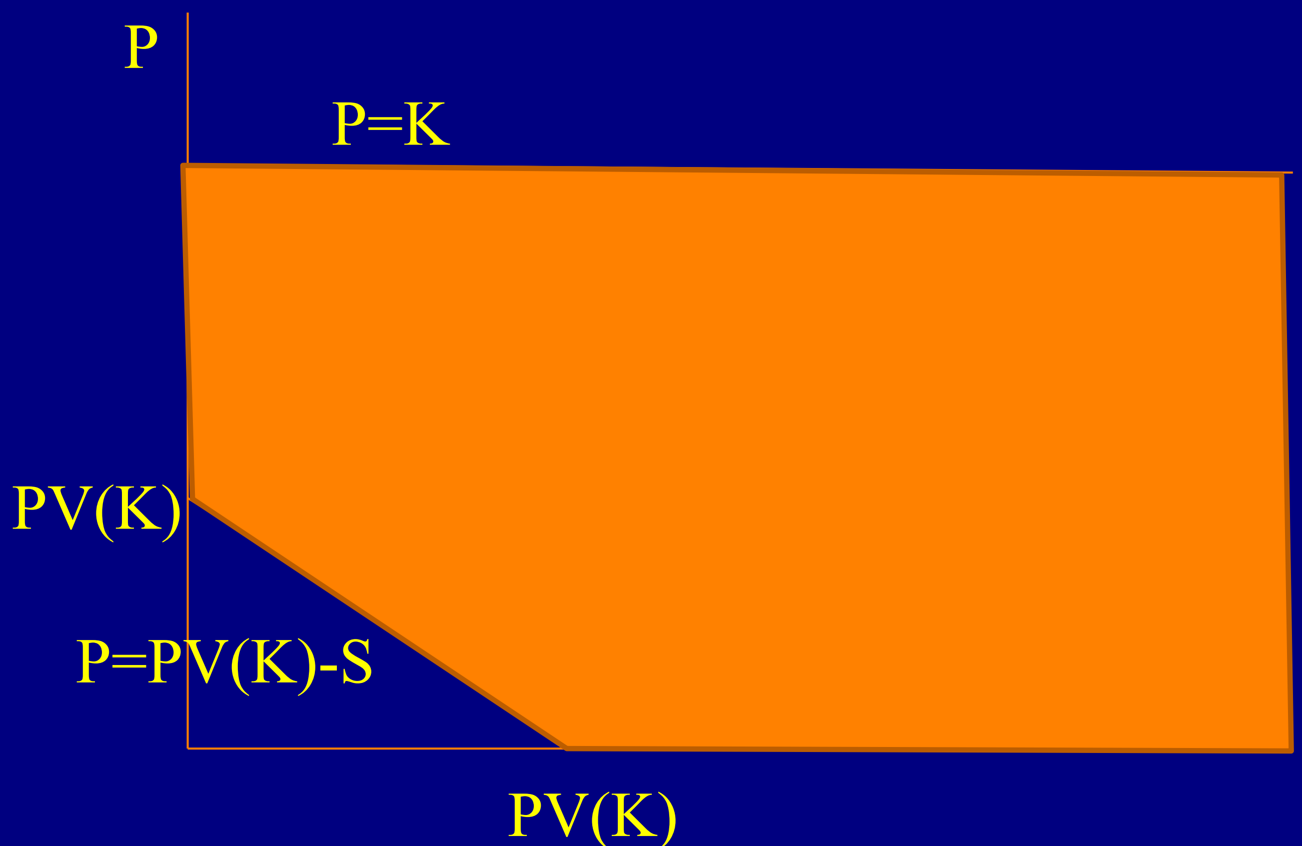
- because the stock price S will drop by the amount of the dividend when the stock goes ex-dividend, i.e., the purchaser of the stock after the ex-dividend date will not receive the dividend payment

Therefore, it is never optimal to exercise an American call option except at expiration or possibly just before the ex-dividend date

A call option is “worth more alive than dead”

Arbitrage Restrictions on Put Prices

1) $P \geq 0$
2) $P \leq K$
3) $P \geq PV(K) - S$
4) $P \geq K - S$
(American put only)

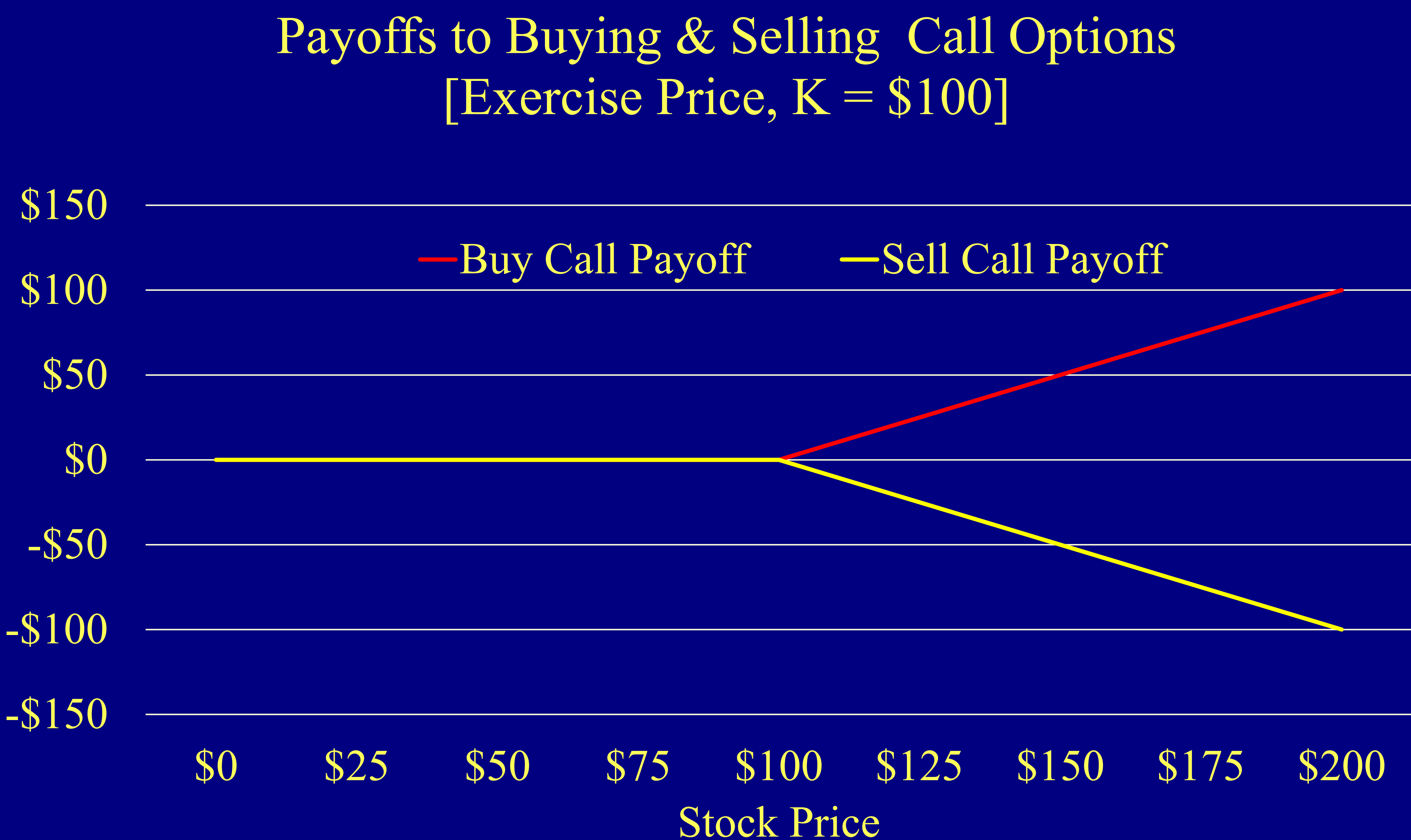


Payoff Diagrams for Contingent Claims

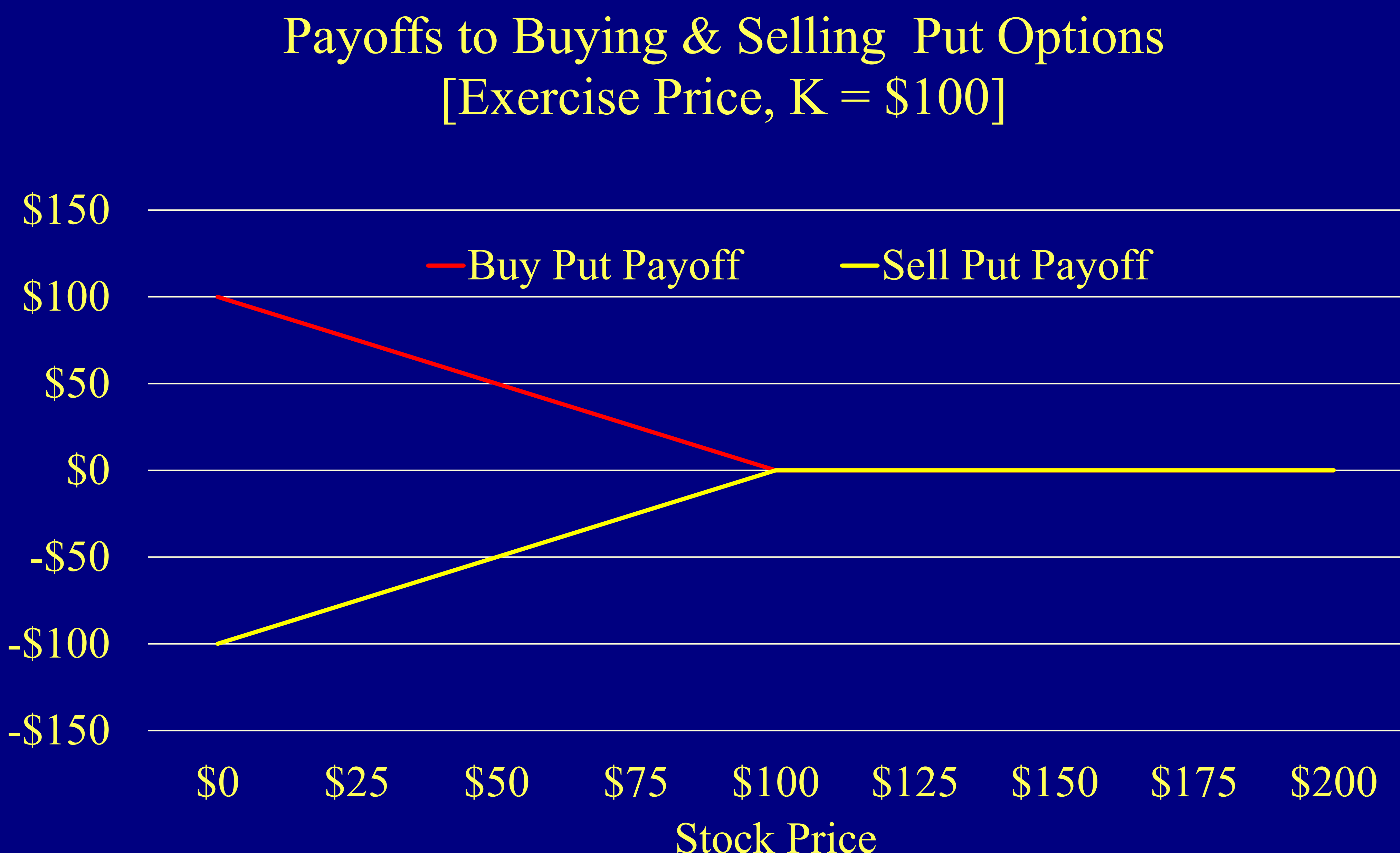
Shows relation between $Payoff and Stock Price for claims with an exercise price, K=$100

- Ignores cost of buying/selling the contingent claims/options
- Ignores transactions costs
- Useful for seeing relations among different contracts
  - E.g., if two different contracts have the same payoffs, they should have the same value/price

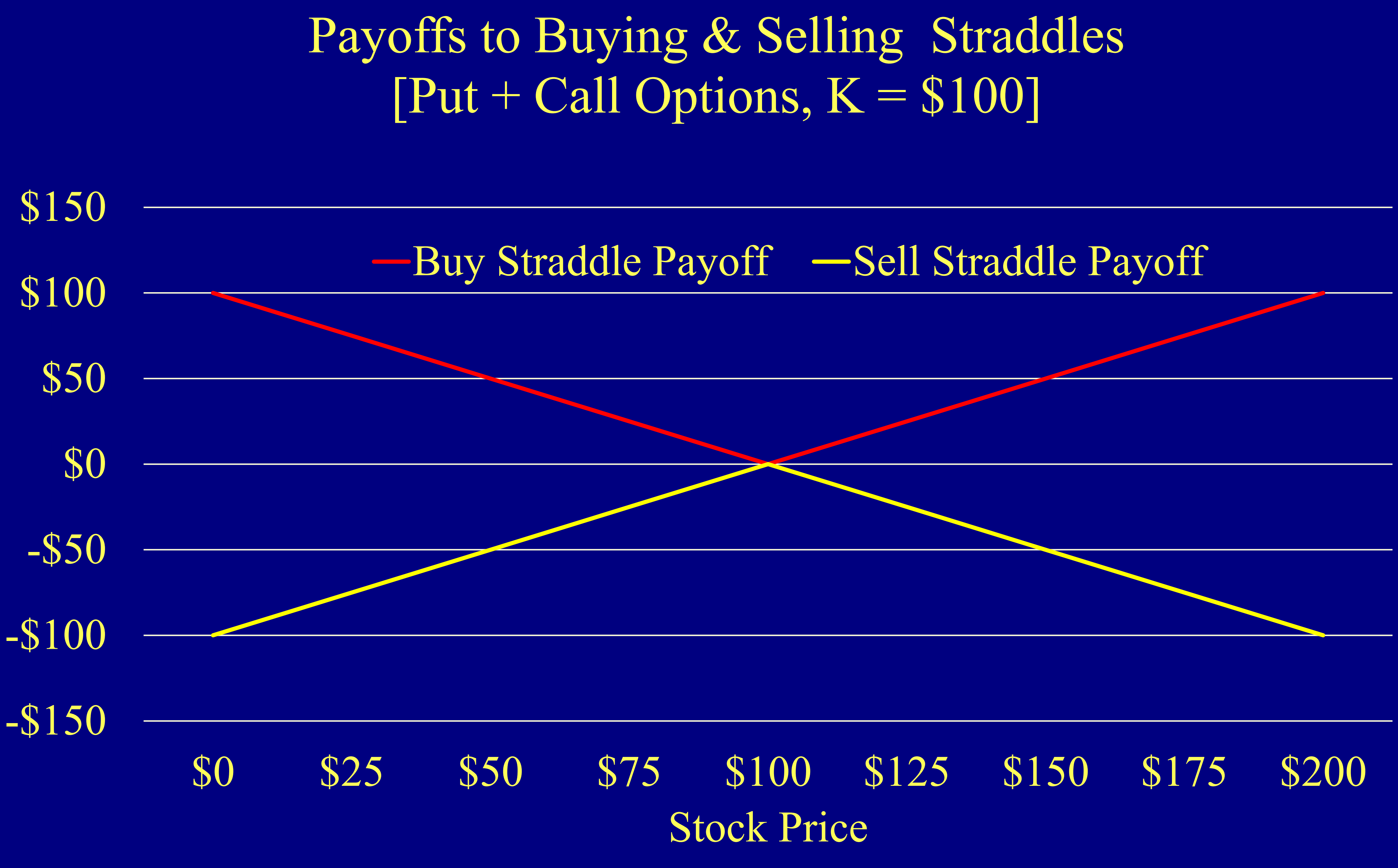
Payoff Diagram for Call



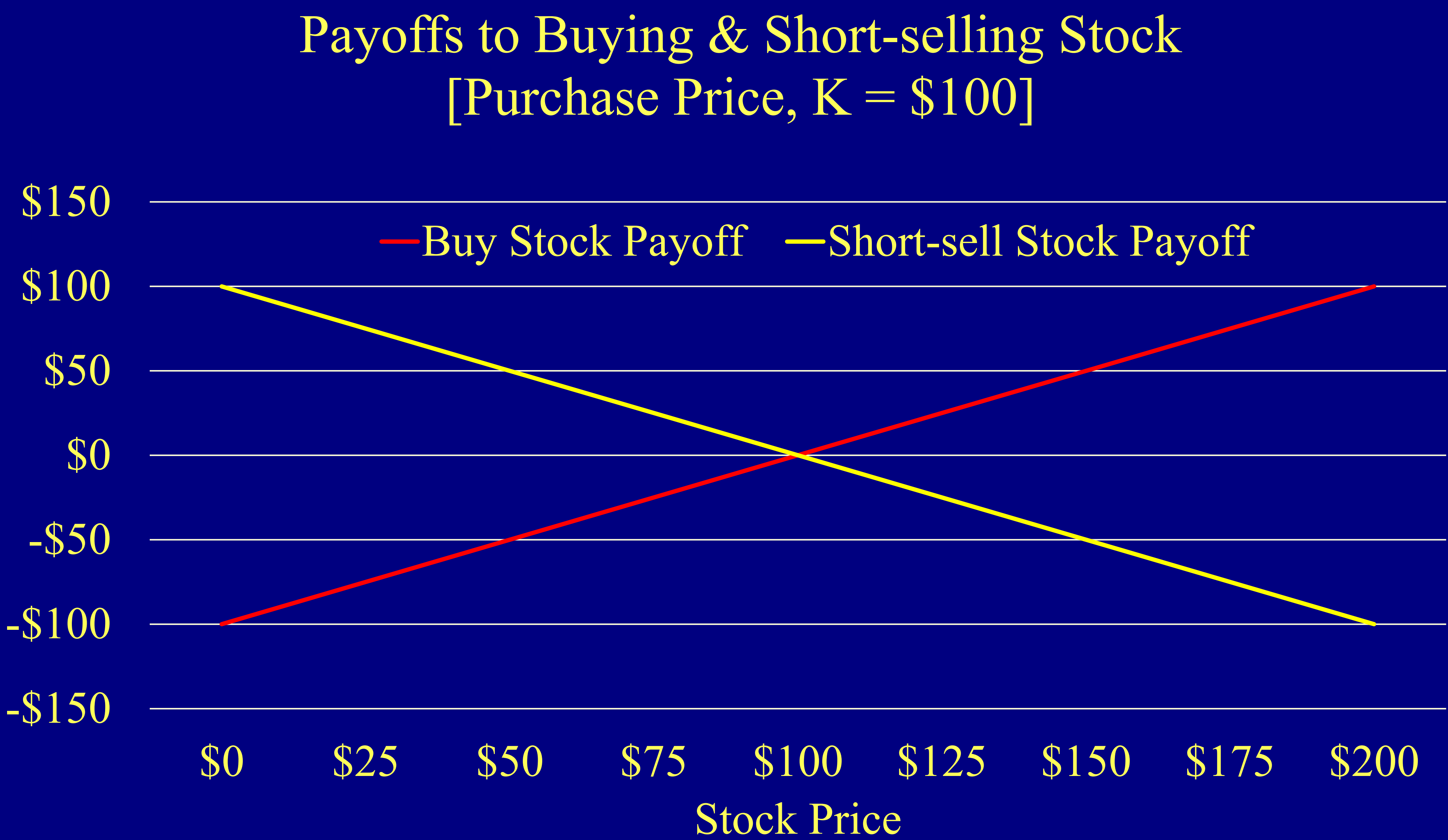
Payoff Diagram for Put



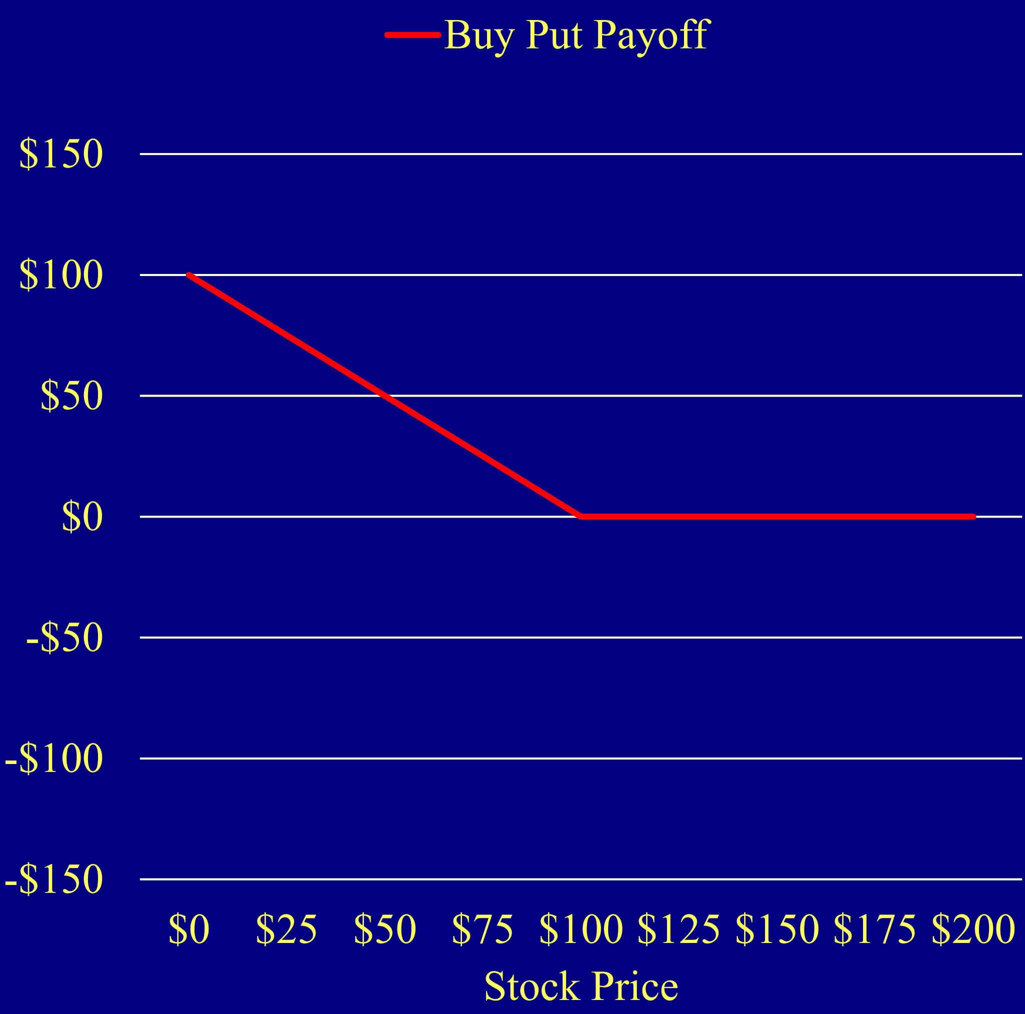
Payoff Diagram for Straddles

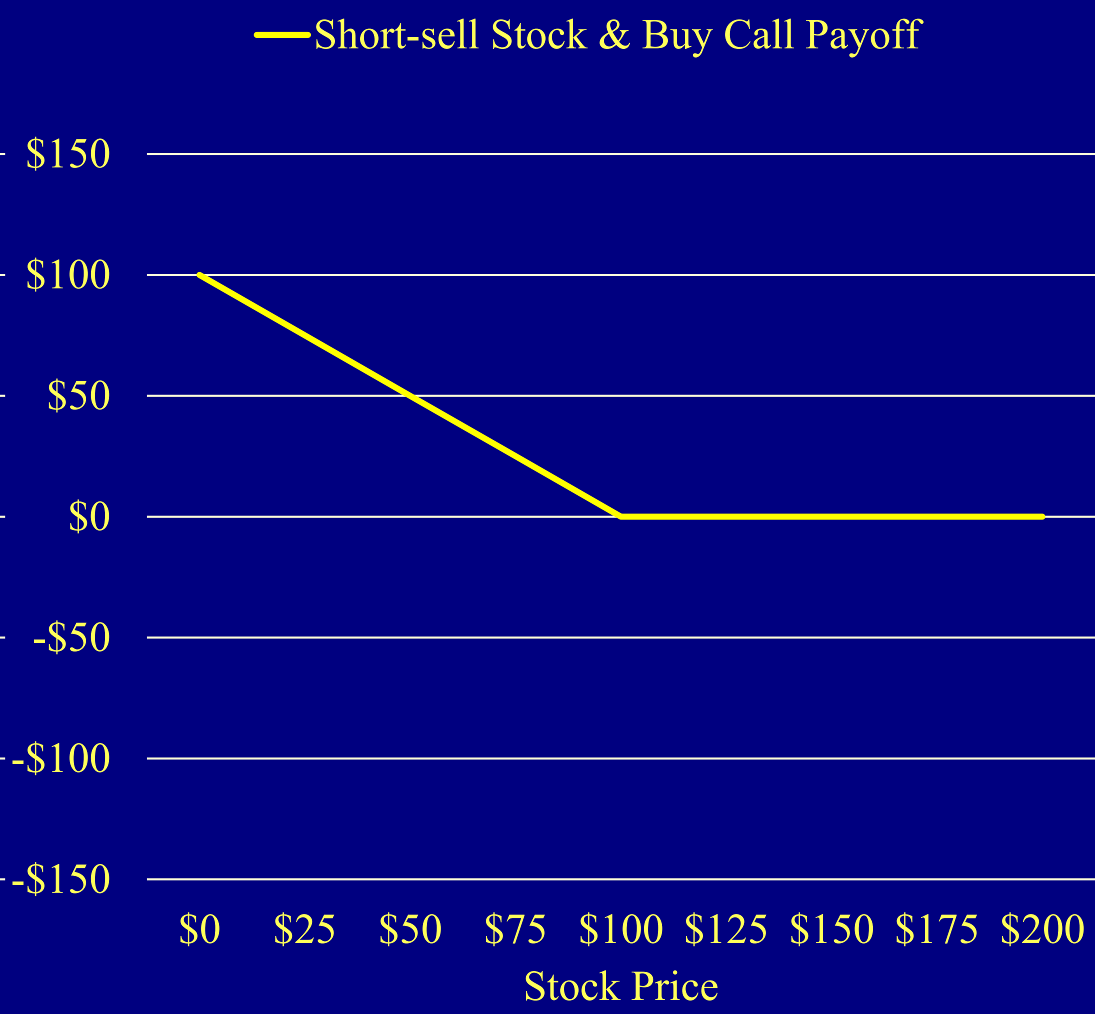


Payoff Diagram for Buying and Short-selling Stock

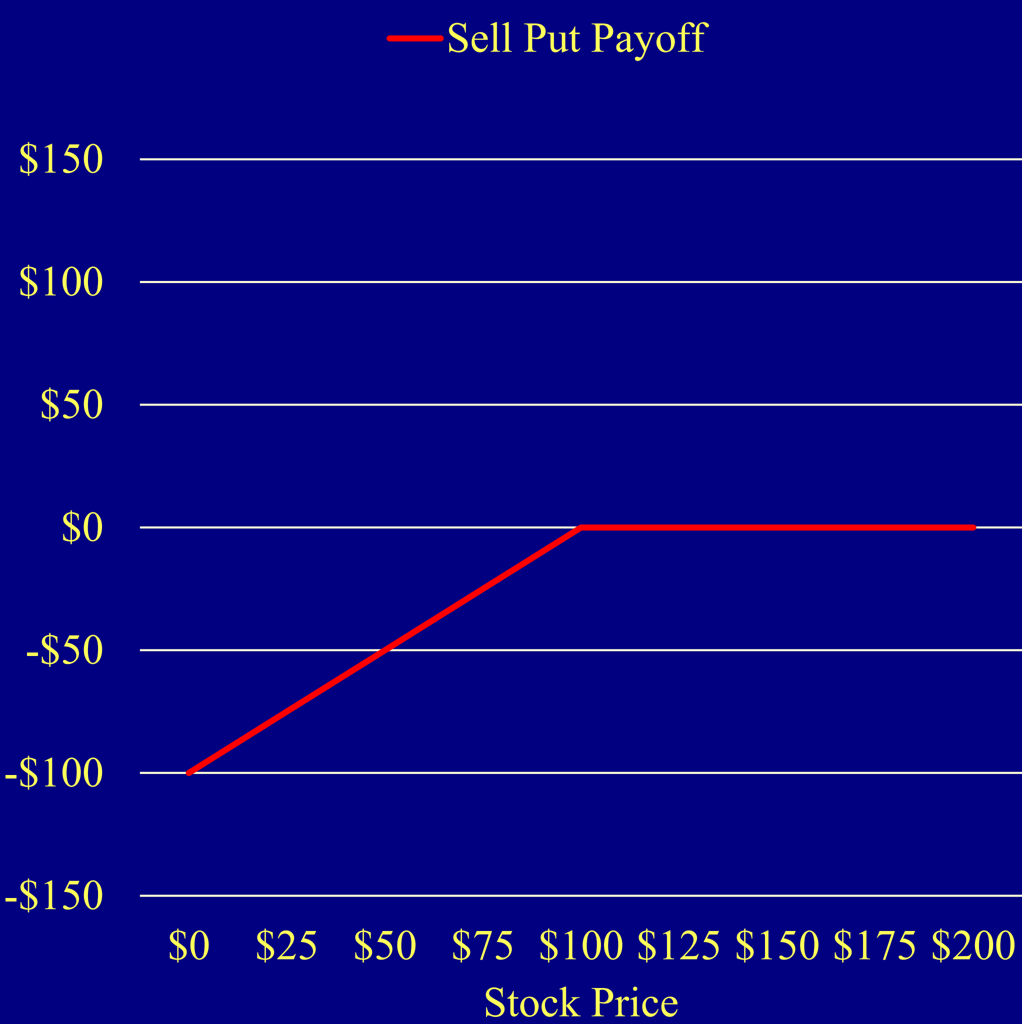


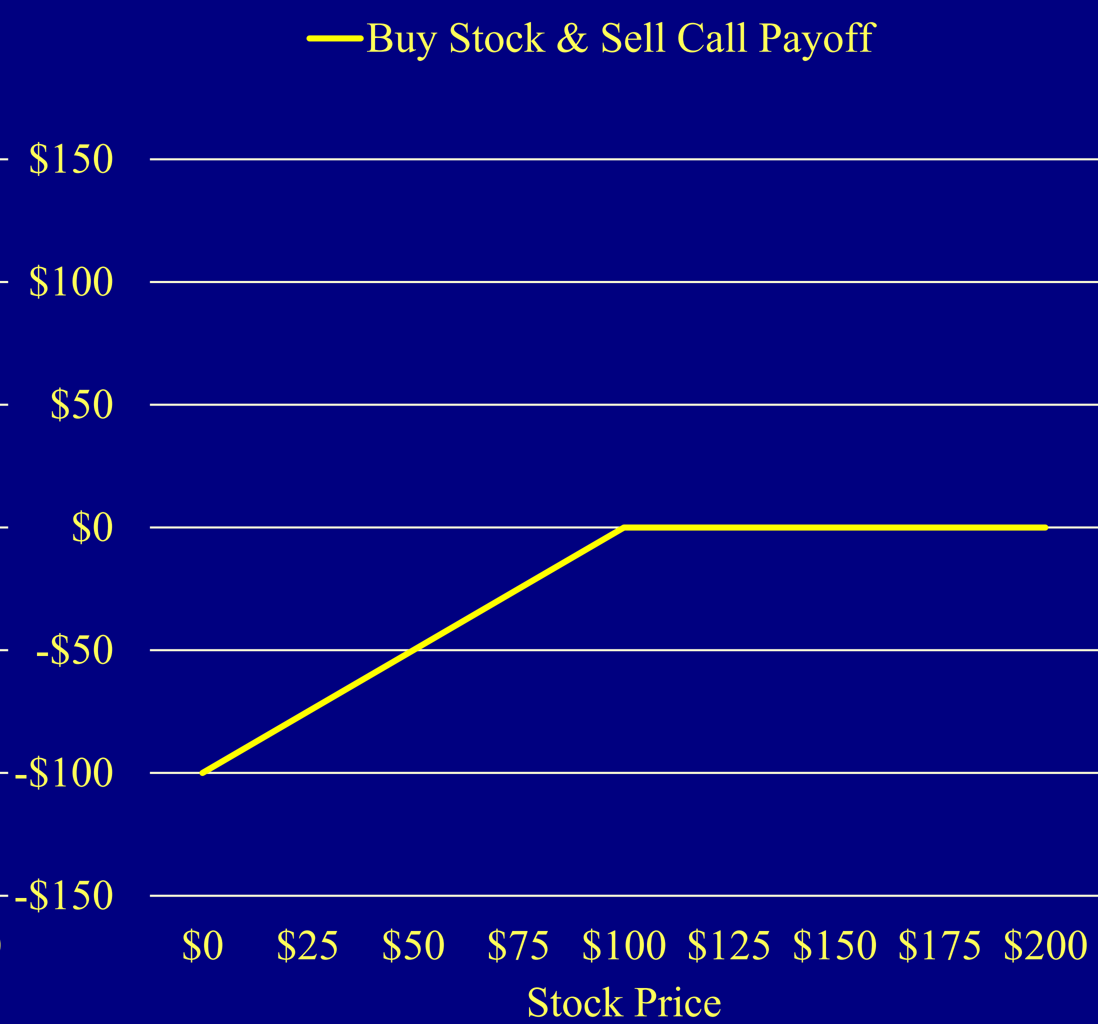
Payoff Diagram for Buying a Put vs. Short-selling Stock and Buying a Call



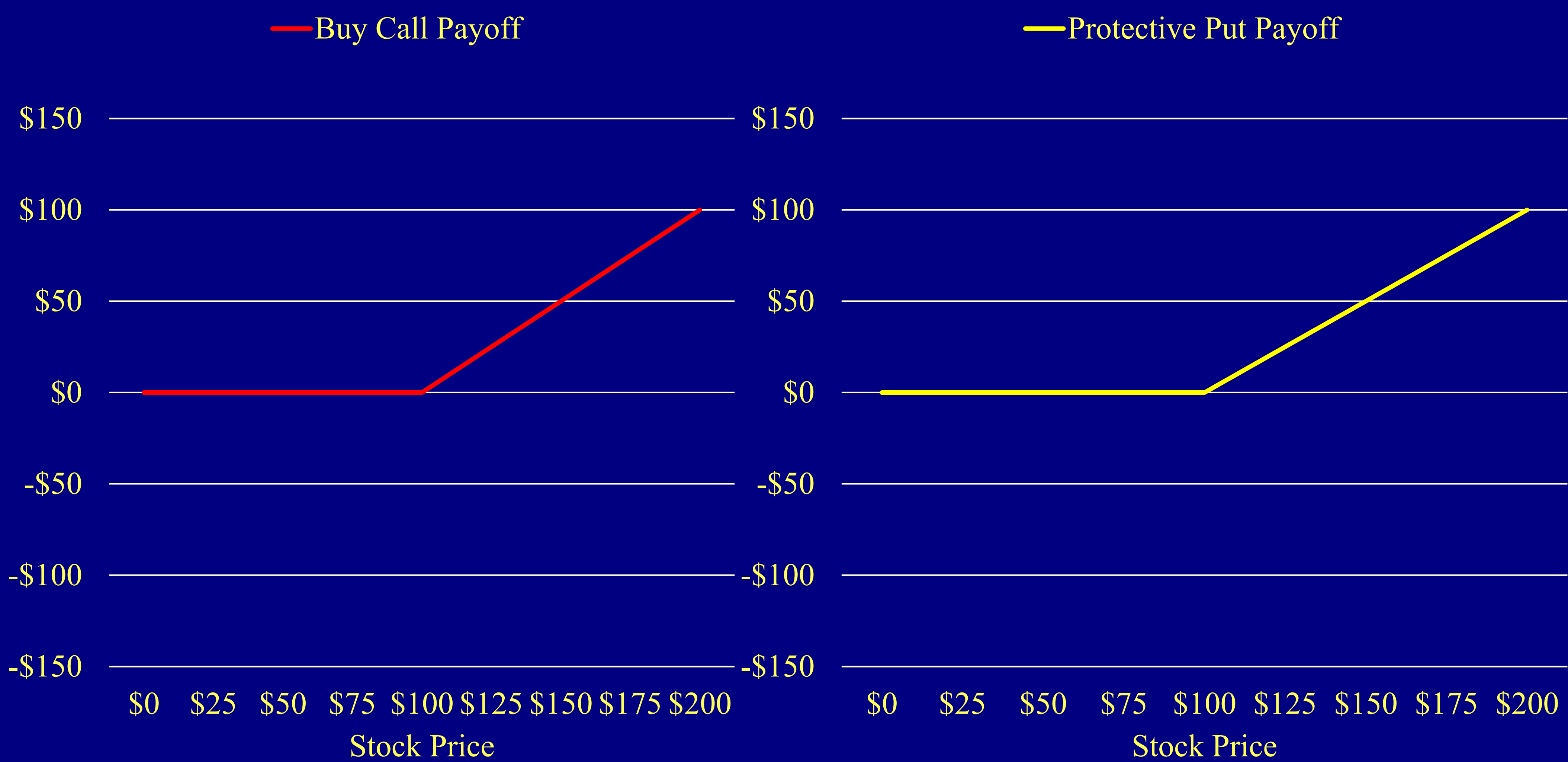


Payoff Diagram for Buying Stock and Selling a Call vs. Selling a Put





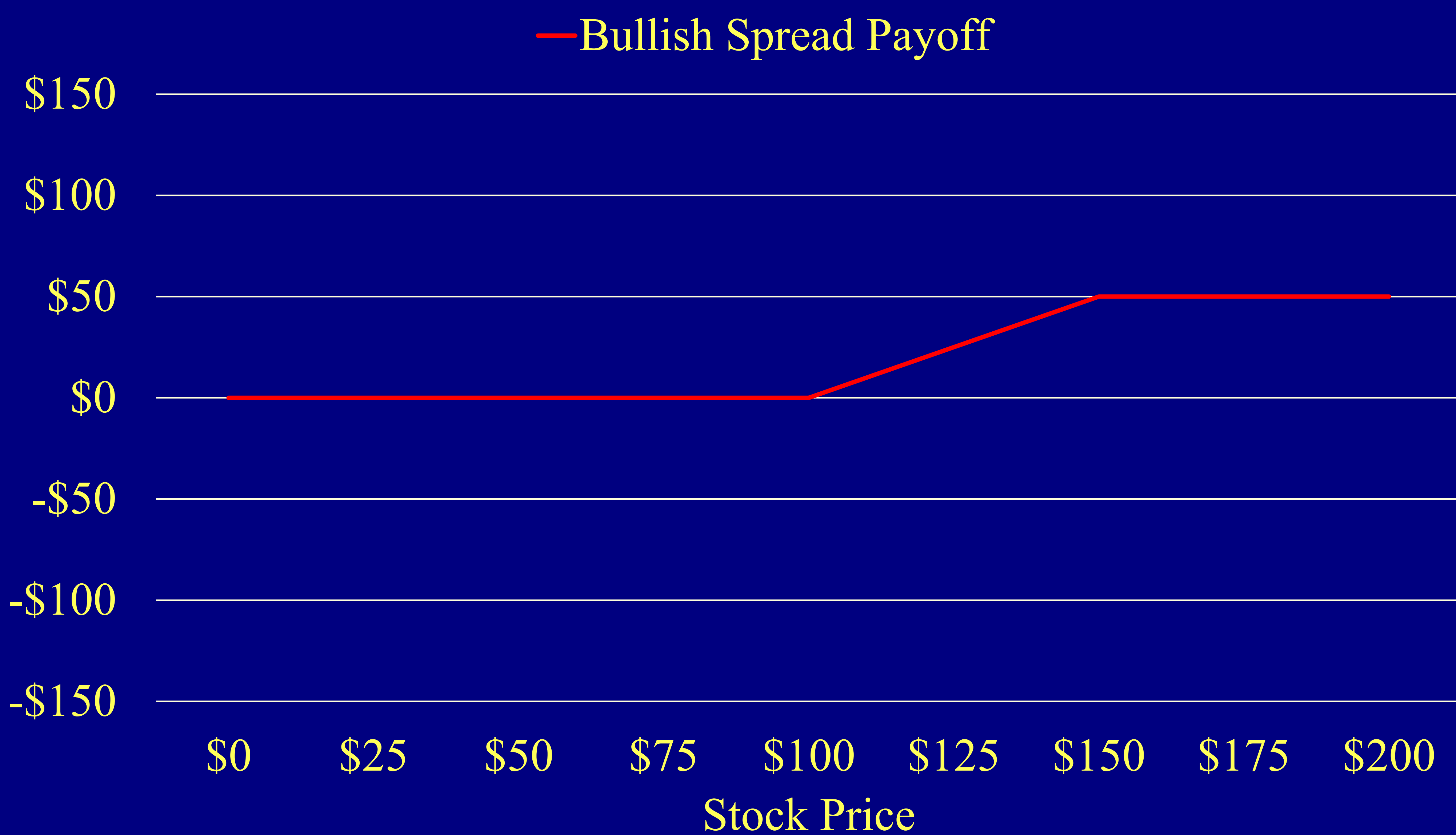
Payoff Diagram for Buying Stock and Selling a Put vs. Buying a Call



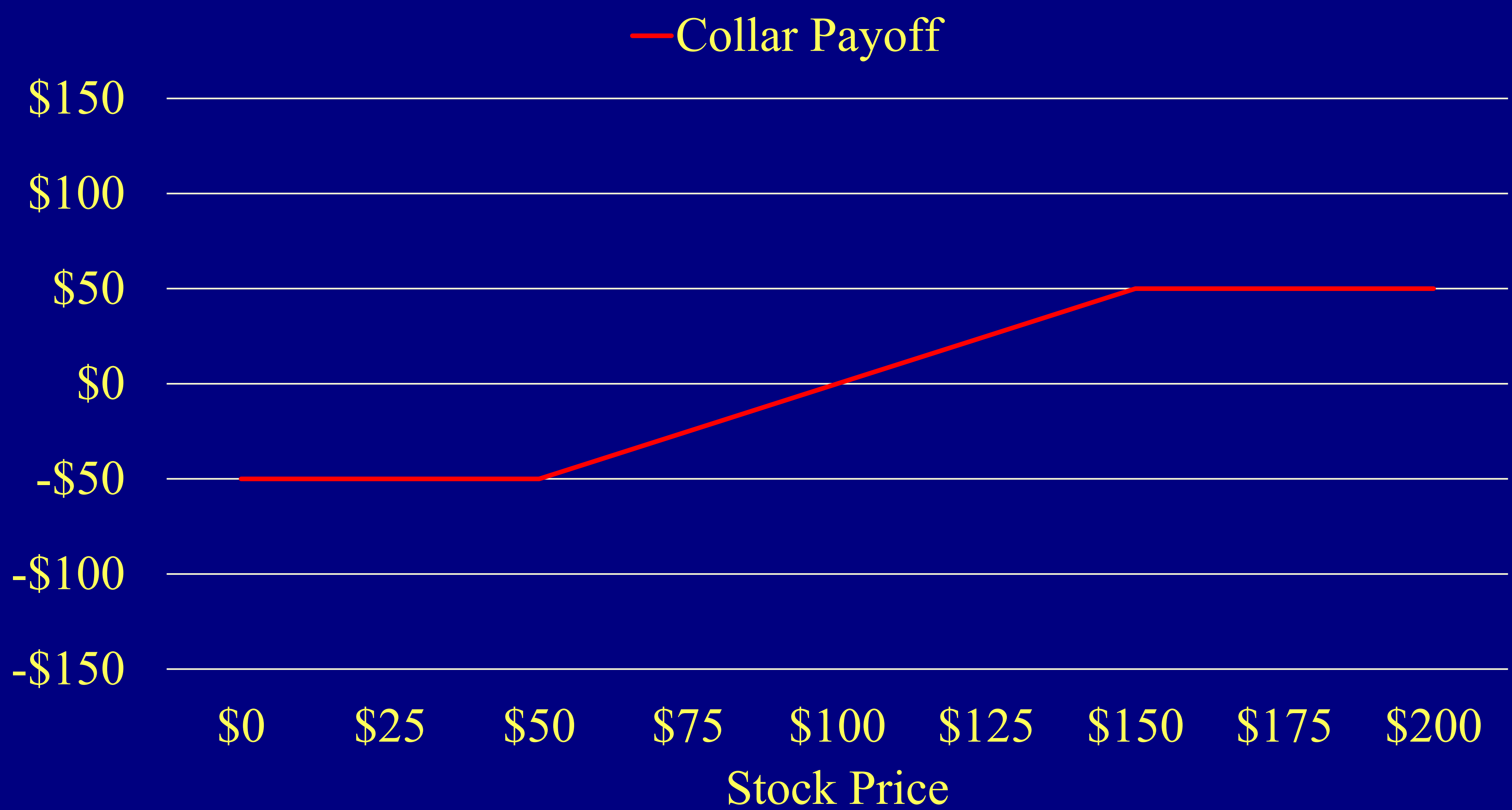
Payoffs Add Up:
Useful for Pricing Contingent Claims

- Put-Call Parity is nothing more than the observation that buying a put is equivalent to short-selling the stock and buying a call
  - Invest the net proceeds in a risk-free bond earning the interest rate r
- You can combine basic options with stocks and risk-free bonds to create any payoff structure you like
  - Presumably the market will price it "fairly"
    - i.e., you will be correctly compensated for the risk you choose to bear

Payoff Diagram for Bullish Spread: Buying at-the-money Call and Sell out-of-the-money Call



Payoff Diagram for Collar: Buy Stock and Sell out-of-the-money Call and Buy out-of-the-money Put



Intuition Behind the Black-Scholes Model

- It is possible to create a portfolio of stocks and bonds that has the exact payoff as a call option over a very short period of time
- Since the stock and bond portfolio and the call option have the same payoffs, they must have the same price or there would be arbitrage opportunities
- Thus, we can value options by identifying this replicating portfolio of stocks and bonds and use the directly observable prices of the stocks and bonds

The Black-Scholes Model: A Simple Example

Assume a call option is available with K = $50

S_{T-1} = $ 50

S_T = either $100
or $25

r = 1.25

What is the value of the call option?

The Black-Scholes Model: A Simple Example

Consider the following portfolio:

		T	
	T-1	S_T = \$25	S_T = \$100
Write 3 calls	3C	0	-150
Buy 2 shares	-100	50	200
Borrow \$40	40	-50	-50
Total	0	0	0

No arbitrage implies that 3C – 100 + 40 = 0
or C = \$20

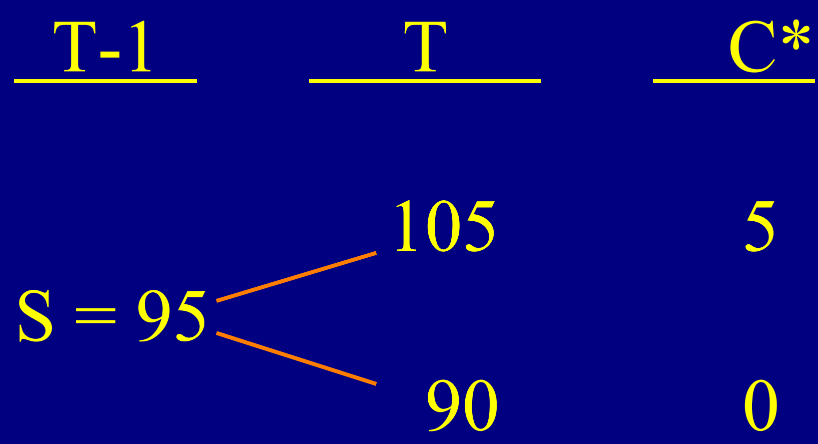
The Black-Scholes Model: A Simple Example

We were able to value the call option in this case because we were able to find a stock and bond portfolio (buy 2/3 of a share and borrow $13.33) that had the same payoff as the call option over this period

Let's try to generalize this reasoning

The Black-Scholes Model:
A Simple Example

Suppose the riskless interest rate is 5%. What is the price of a call option with an exercise prices of $100?



Create a portfolio of Δ shares of stock and B dollars of bonds where Δ and B are chosen so that the stock and bond portfolio has the same payoffs as the call option

The Black-Scholes Model: A Simple Example

T-1	T	C*
	105	5
S = 95		
	90	0

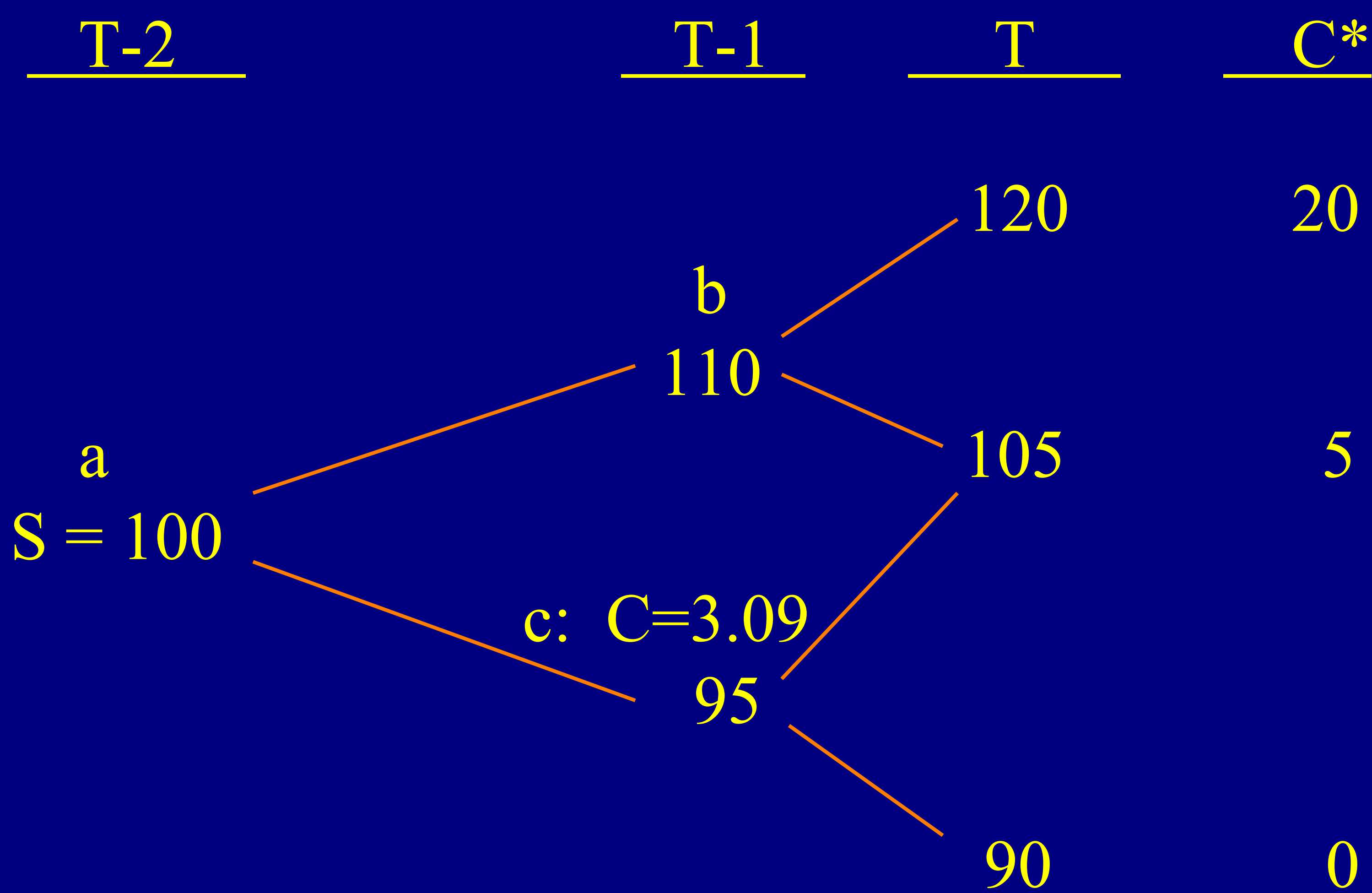
$$105\,\Delta + 1.05\,B = 5$$
$$90\,\Delta + 1.05\,B = 0$$

$$15\,\Delta = 5;\ \Delta = 0.3333$$

$$90\,(0.3333) + 1.05\,B = 0$$
$$B = -28.57$$

$$C = S\,\Delta + B$$
$$= 95\,(0.3333) - 28.57$$
$$= 3.09$$

The Black-Scholes Model: A Simple Example



The Black-Scholes Model: A Simple Example

b: $120\ \Delta + 1.05\ B = 20$

$105\ \Delta + 1.05\ B = 5$

$\Delta = 1$

$105 + 1.05\ B = 5$

$B = -95.24$

$C = S\ \Delta + B$

$= 110\ (1) - 95.24$

$= 14.76$

The Black-Scholes Model: A Simple Example

a: $110\,\Delta + 1.05\,B = 14.76$

$95\,\Delta + 1.05\,B = 3.09$

$\Delta = 0.778$

$95\,(0.778) + 1.05\,B = 3.09$

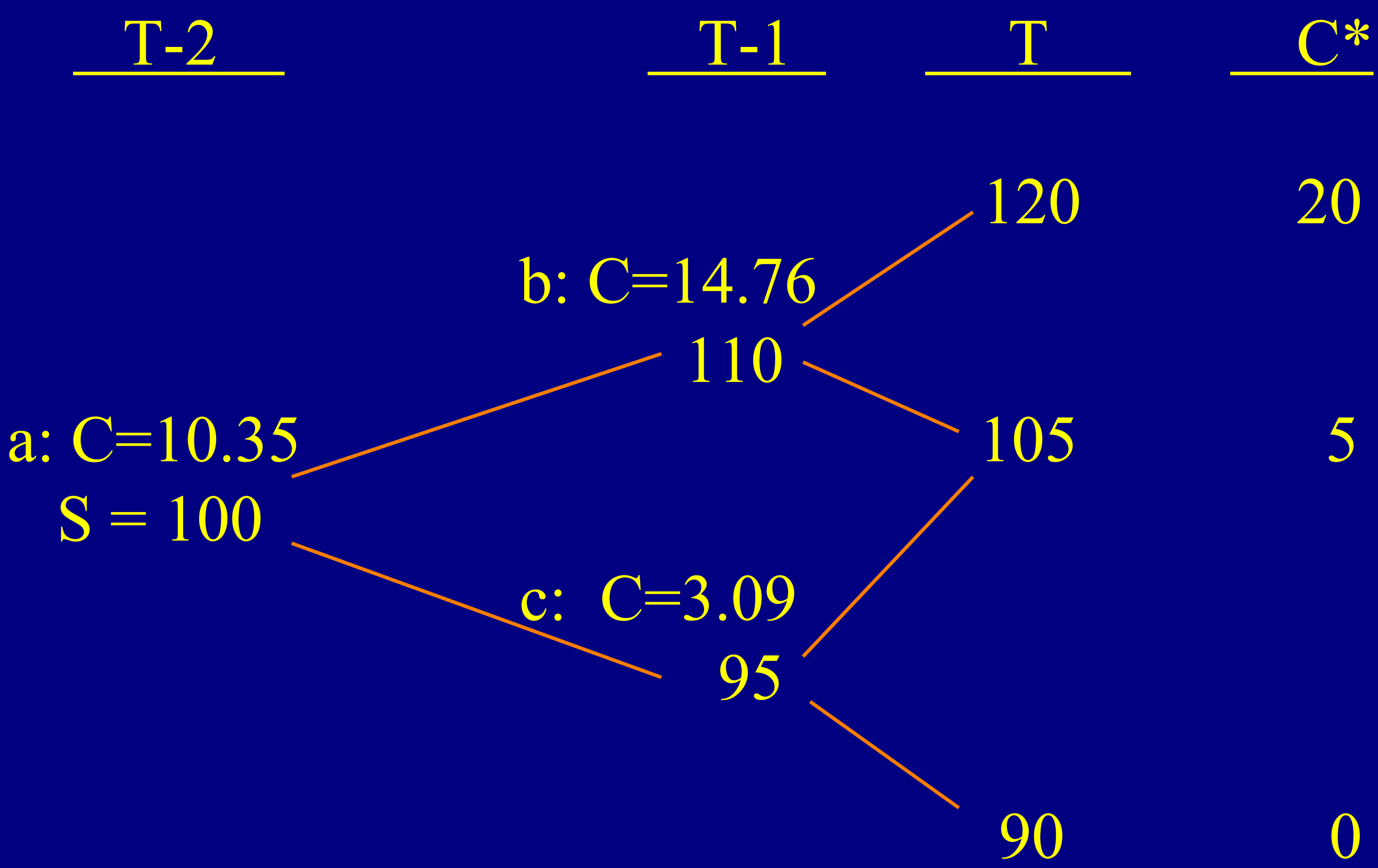
$B = -67.45$

$C = S\,\Delta + B$

$= 100\,(0.778) - 67.45$

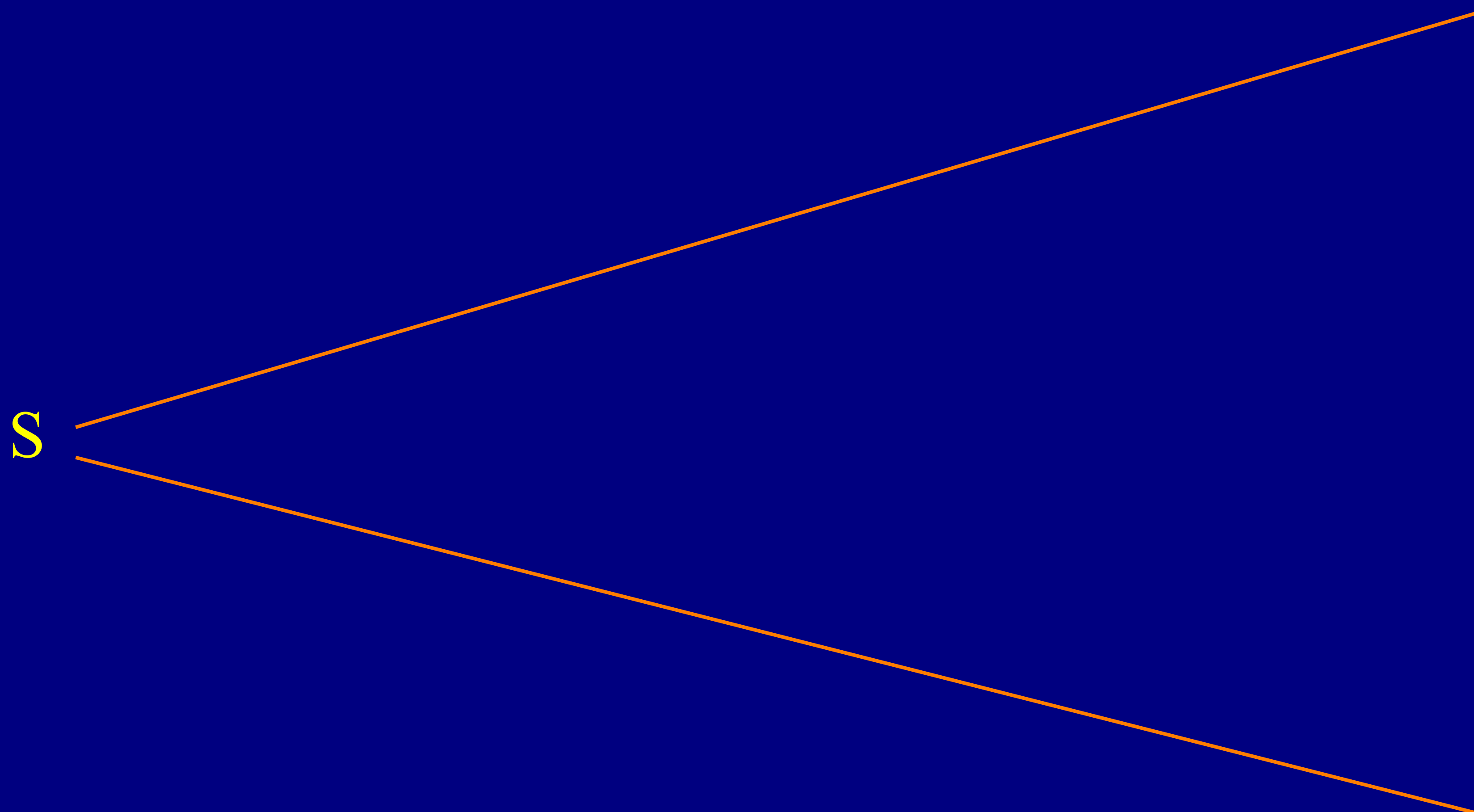
$= 10.35$

The Black-Scholes Model: A Simple Example



Derivation of the Black-Scholes Model

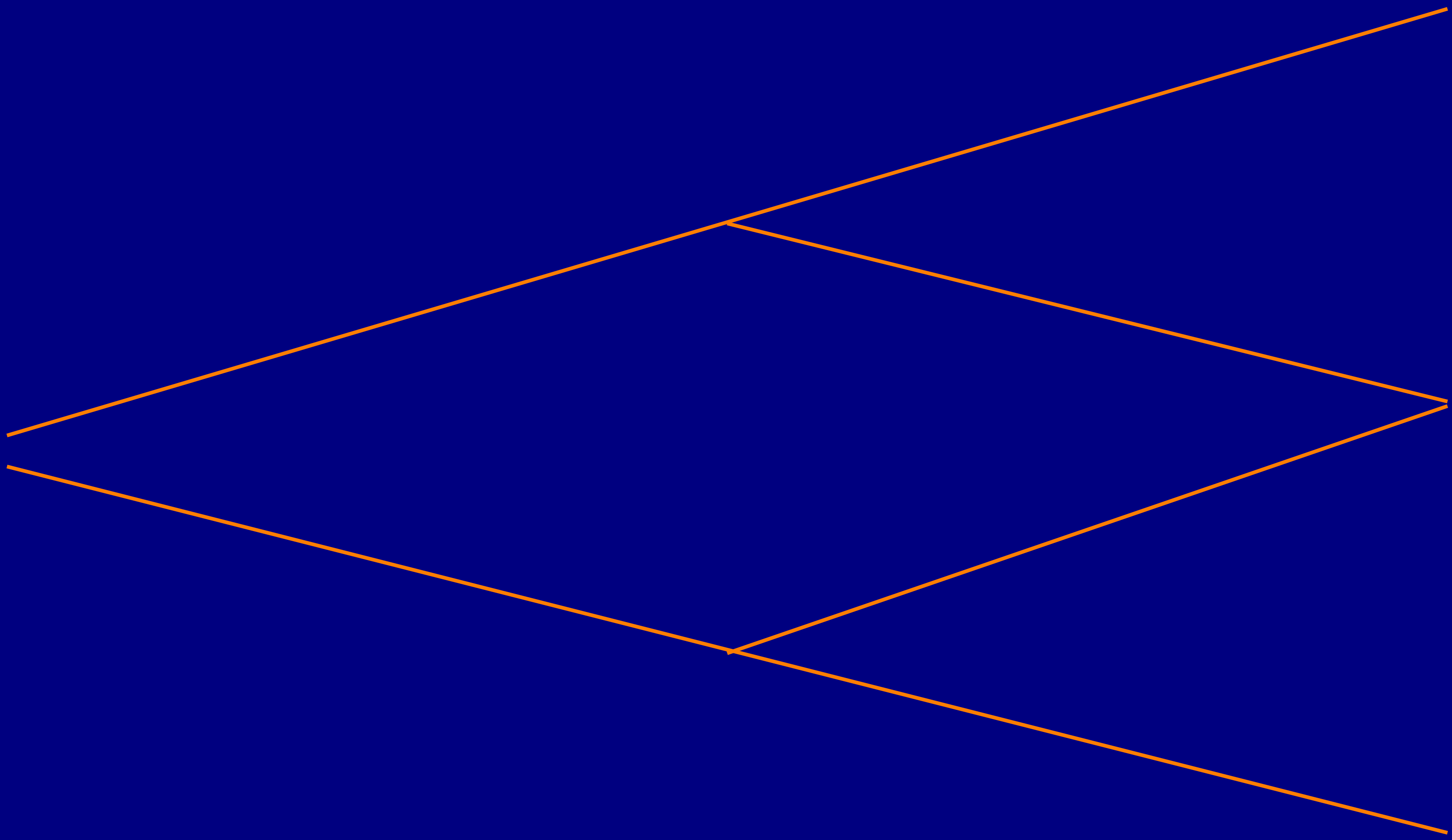
Consider what happens as you take a fixed interval of time and divide it into more subintervals



Derivation of the Black-Scholes Model

Consider what happens as you take a fixed interval of time and divide it into more subintervals

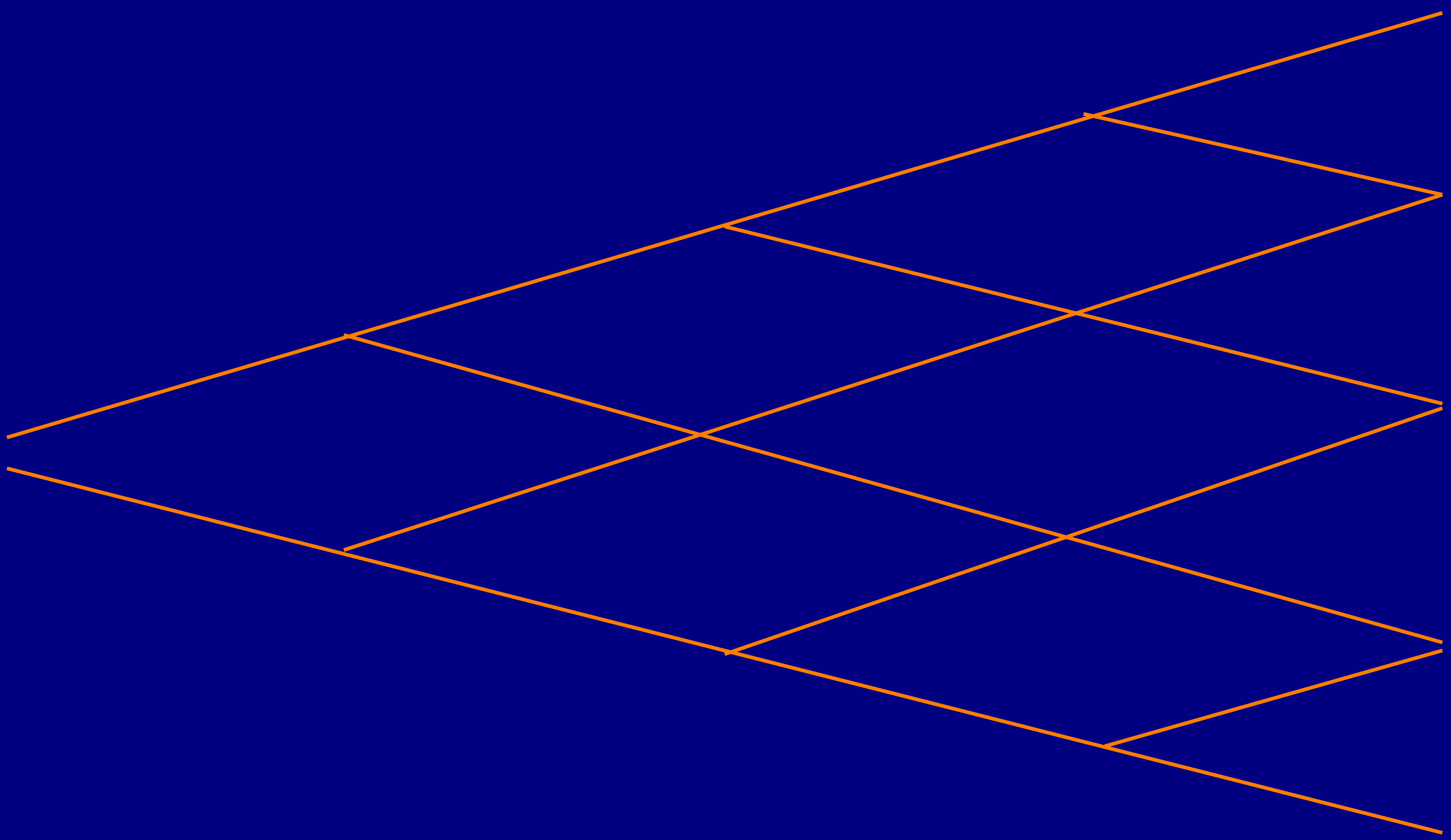
S



Derivation of the Black-Scholes Model

Consider what happens as you take a fixed interval of time and divide it into more subintervals

S



Derivation of the Black-Scholes Model

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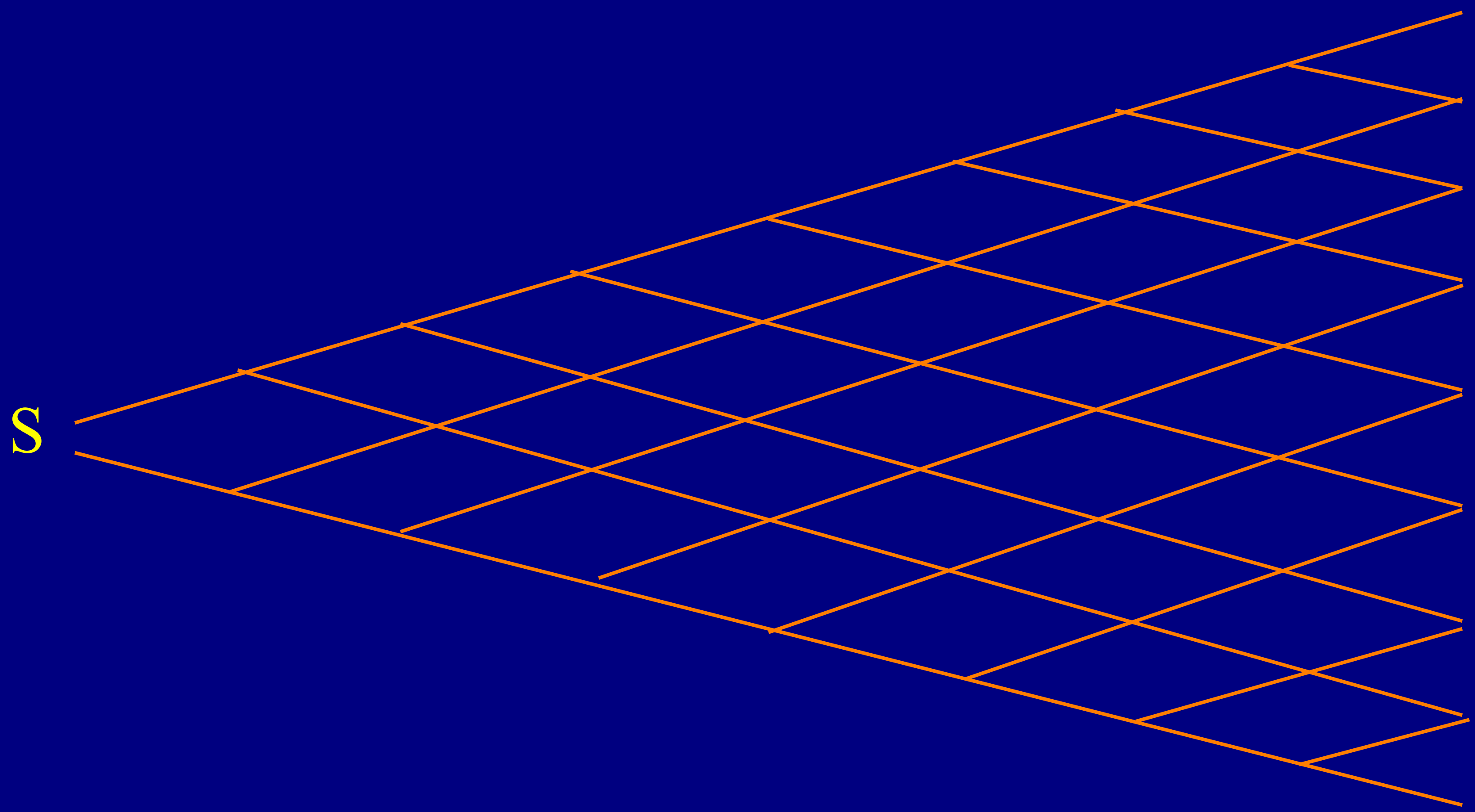


The Black-Scholes Model

Create a hedge portfolio:

$$V_H = S\,Q_S + C\,Q_C$$

$$dV_H = dS\,Q_S + dC\,Q_C \qquad (1)$$

where S is the stock price, C is the call price, and Q_S and Q_C are the amounts invested in S and C

So far, this looks like a standard calculus problem

- The only problem is that C and S are correlated random variables so the standard rules of calculus do not apply

Ito's Lemma

If C = C(S,t) where C and S are random variables, then

$$dC = (\partial C/\ \partial S)\ dS + (\partial C/\ \partial t)\ dt + \tfrac{1}{2}\ (\partial^2 C/\ \partial S^2)\ \sigma^2 S^2\ dt$$

Assumptions needed for Ito's lemma:

- Stock prices are continuous
- Stock prices have no memory
- Option price is a function of the current price, but not a function of the past price path

The Black-Scholes Model: A Risk-free Hedge

$$dV_H = dS\ Q_S + [(\partial C/\partial S)\ dS + (\partial C/\partial t)\ dt + \tfrac{1}{2}(\partial^2 C/\partial S^2)\ \sigma^2 S^2\ dt]\ Q_C$$

Choose Q_S and Q_C so that

$$dS\ Q_S + (\partial C/\partial S)\ dS\ Q_C = 0$$

In other words, V_H is risk-free as long as

$$Q_S / Q_C = -(\partial C/\partial S)$$

The Black-Scholes Model: A Risk-free Hedge

Since V_H is risk-free, its rate of return must be the risk-free interest rate, i.e.,

$$dV_H / V_H = r\,dt$$

If you make the necessary substitutions, you are left with a partial differential equation without any random variables

- the solution to this equation, with the boundary condition C* = max[0, S-K)], is the Black-Scholes option pricing model

The Black-Scholes Model

$C = S\,N\{[\mathit{ln}(S/K) + (r + \sigma^2/2)\,T]/\,\sigma\sqrt{T}\}$ –

$\mathit{exp}(-rT)\,K\,N\{[\mathit{ln}(S/K) + (r - \sigma^2/2)\,T]/\,\sigma\sqrt{T}\}$

Where *ln* is the natural logarithm, *exp* is the exponential function and N{z} is the cumulative normal distribution function

Note *ln*(S/K) = 0, the option is "at the money" (S = K)

When: z = 0, N{z} =.5; z = 2, N{z} =.975;
z = -2, N{z} =.025

σ is the standard deviation of the stock return per unit time, so if either σ or T = 0 (no uncertainty), N{z} = 1

exp(-rT) K is the present value of the exercise price, K

The Black-Scholes Model

Valuing an option with no uncertainty about exercising:

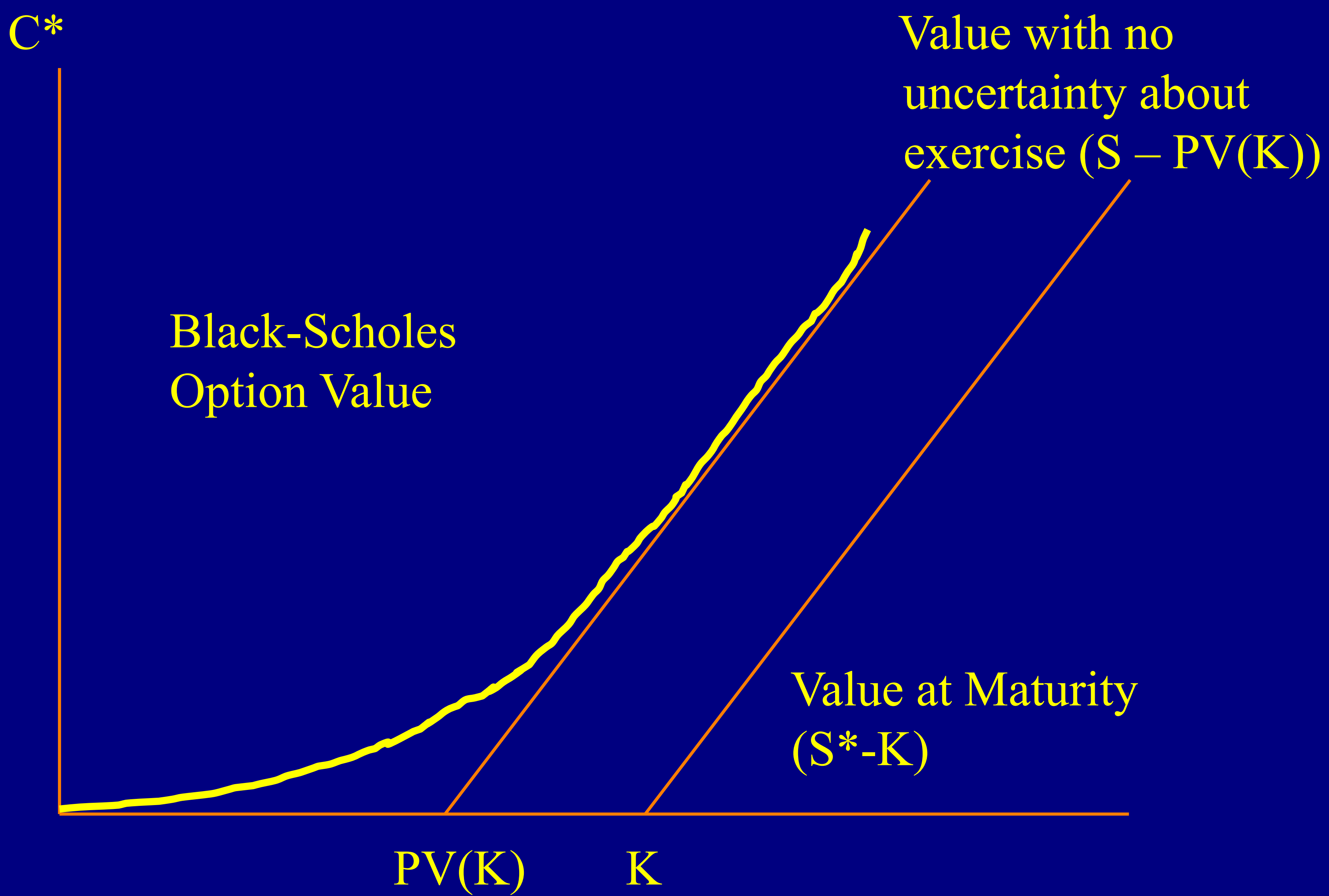
C = PV (C*)

= PV [max(0, S* - K)]

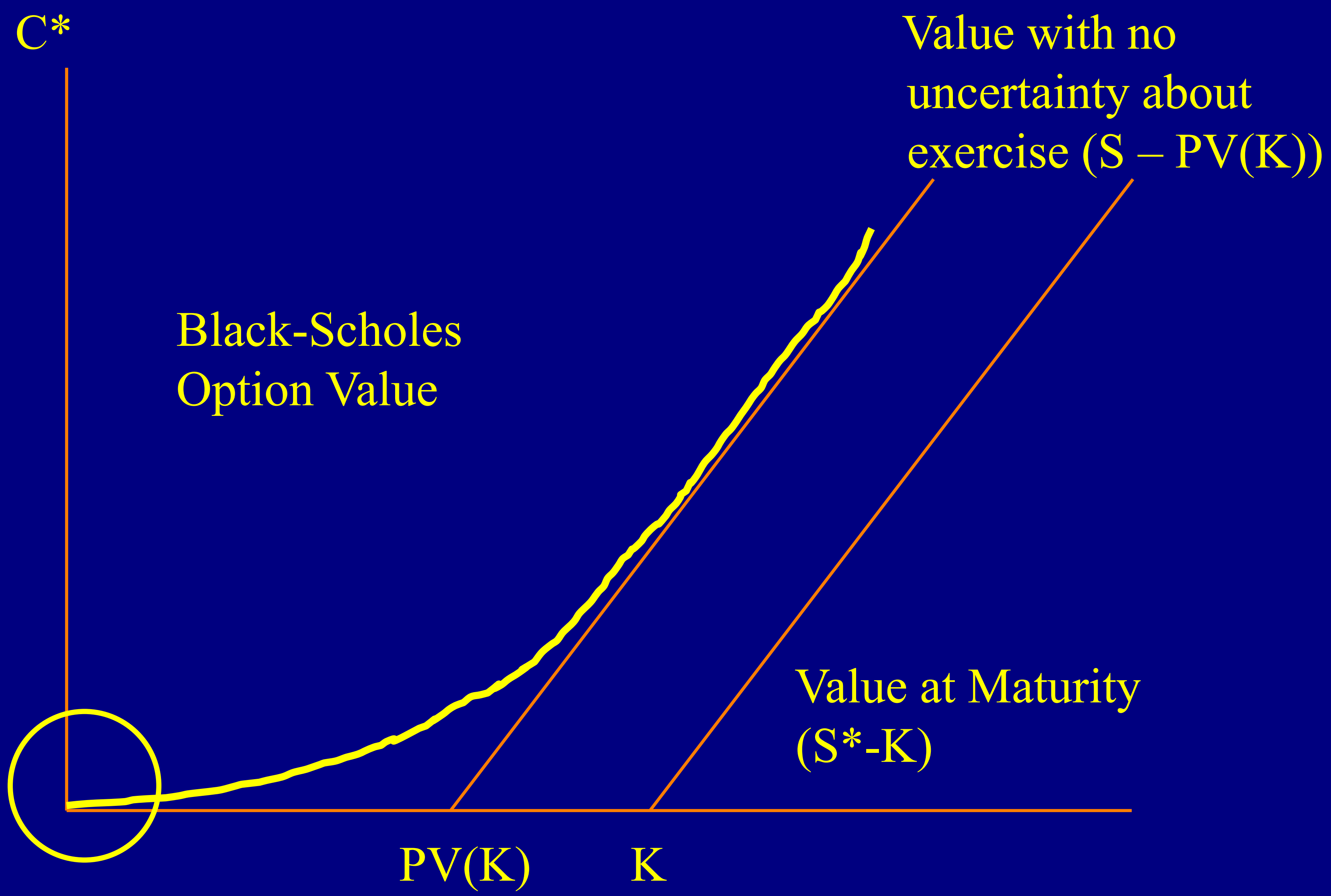
= PV (S* - K) is the option is in the money

= S - *exp*(-rT) K

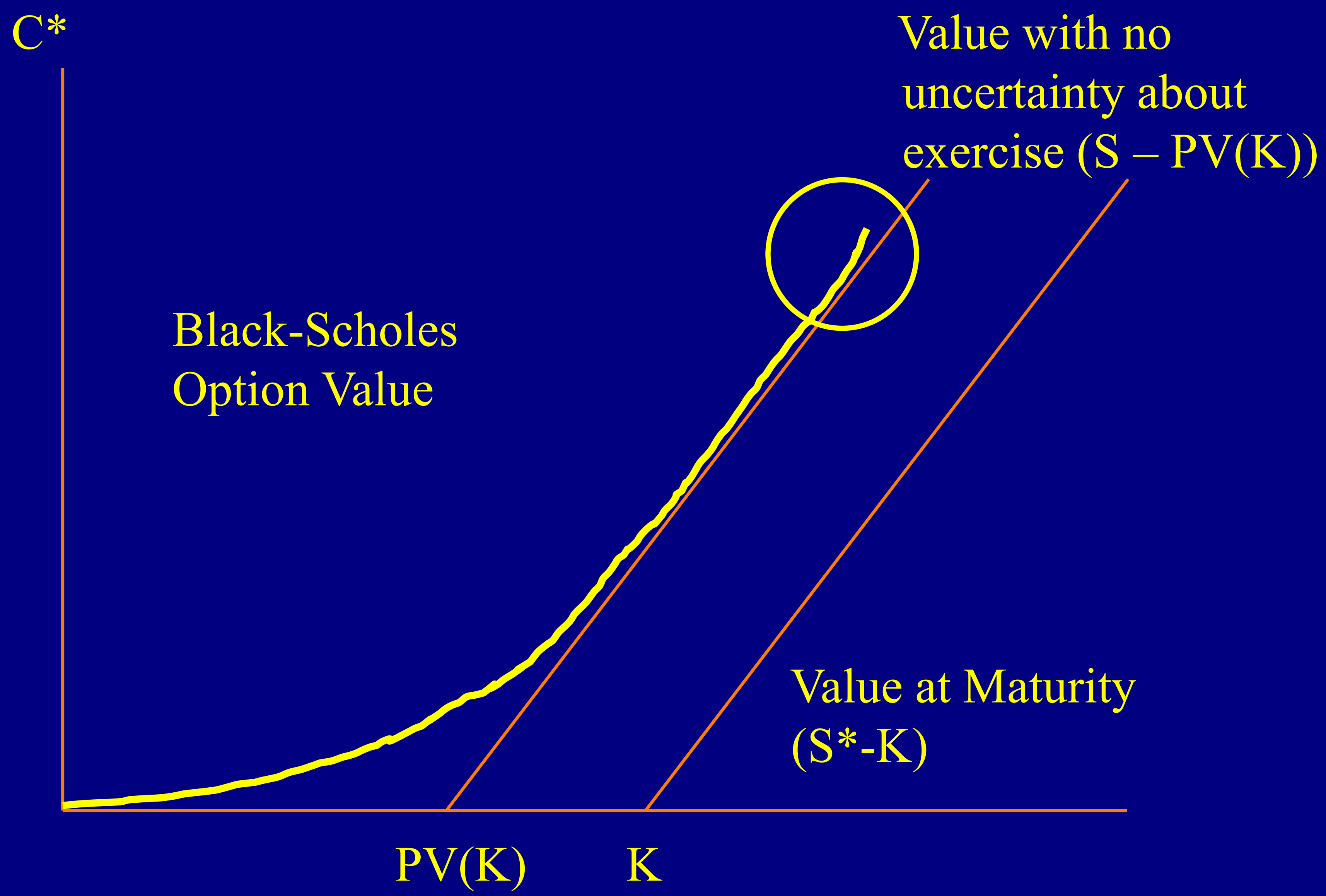
The Black-Scholes Model: Boundary Conditions



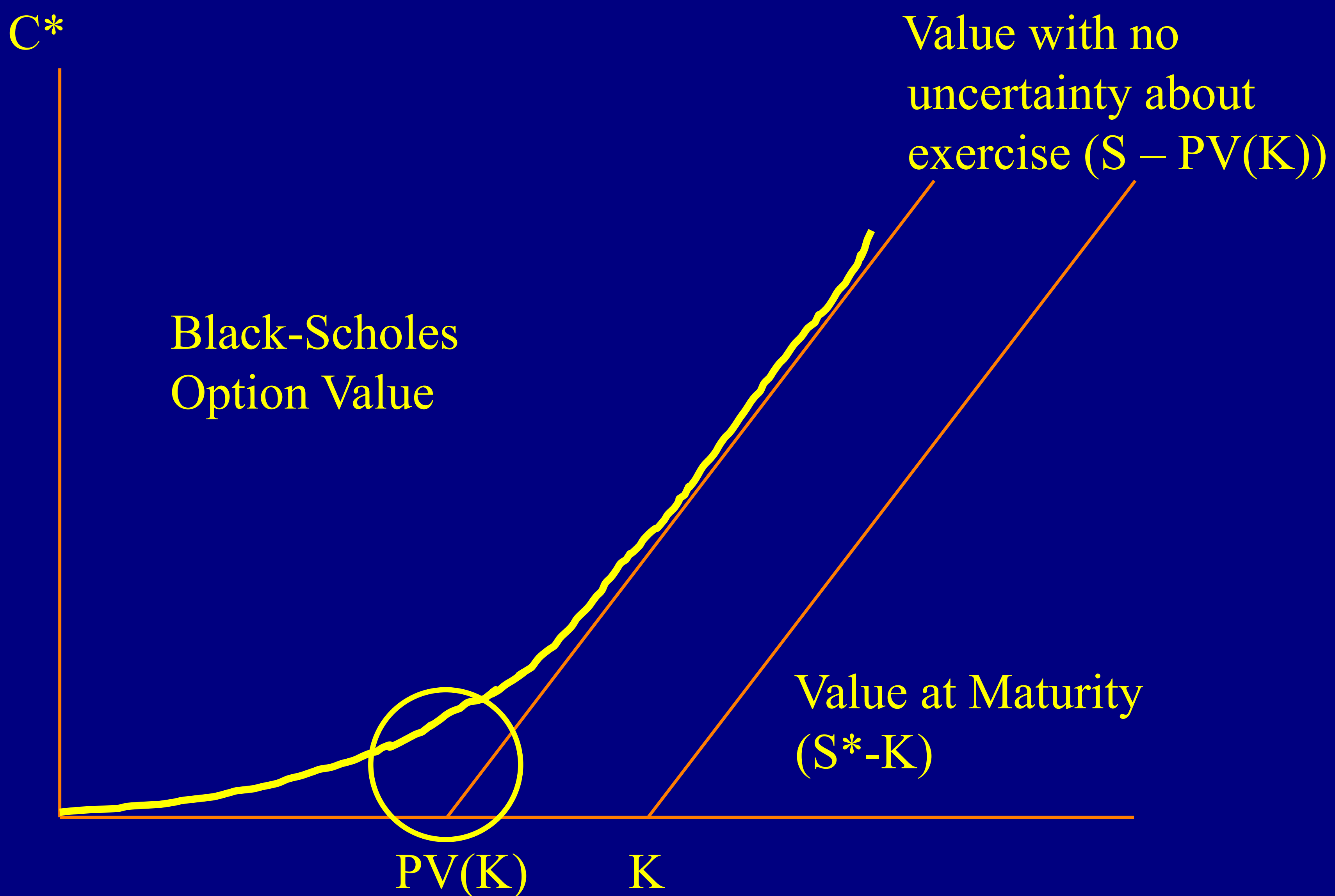
The Black-Scholes Model
Out of the Money: S > C > 0



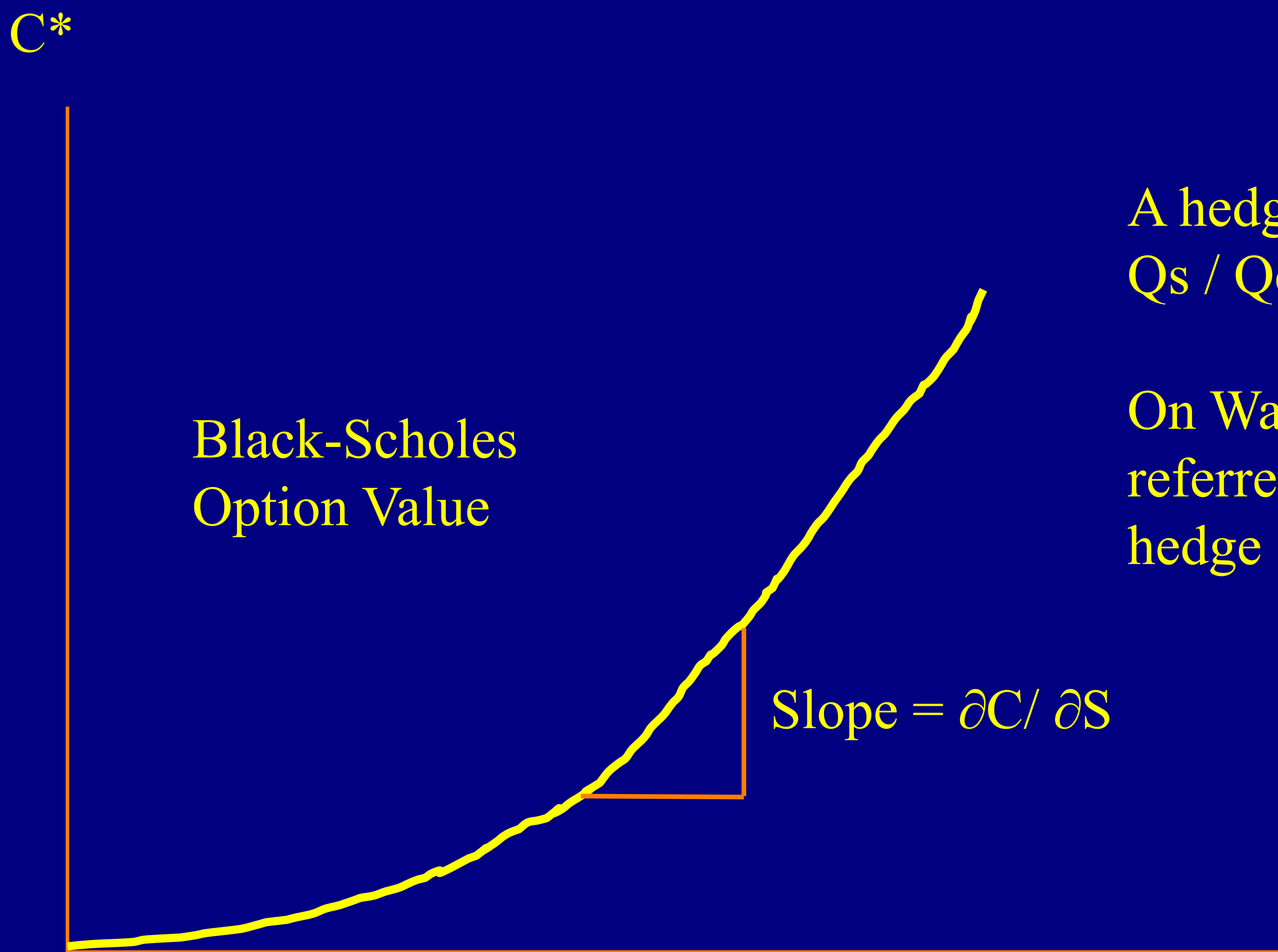
The Black-Scholes Model
In the Money: C > S – PV(K)



The Black-Scholes Model
At the Money Options are Most Valuable



The Black-Scholes Risk-free Hedge



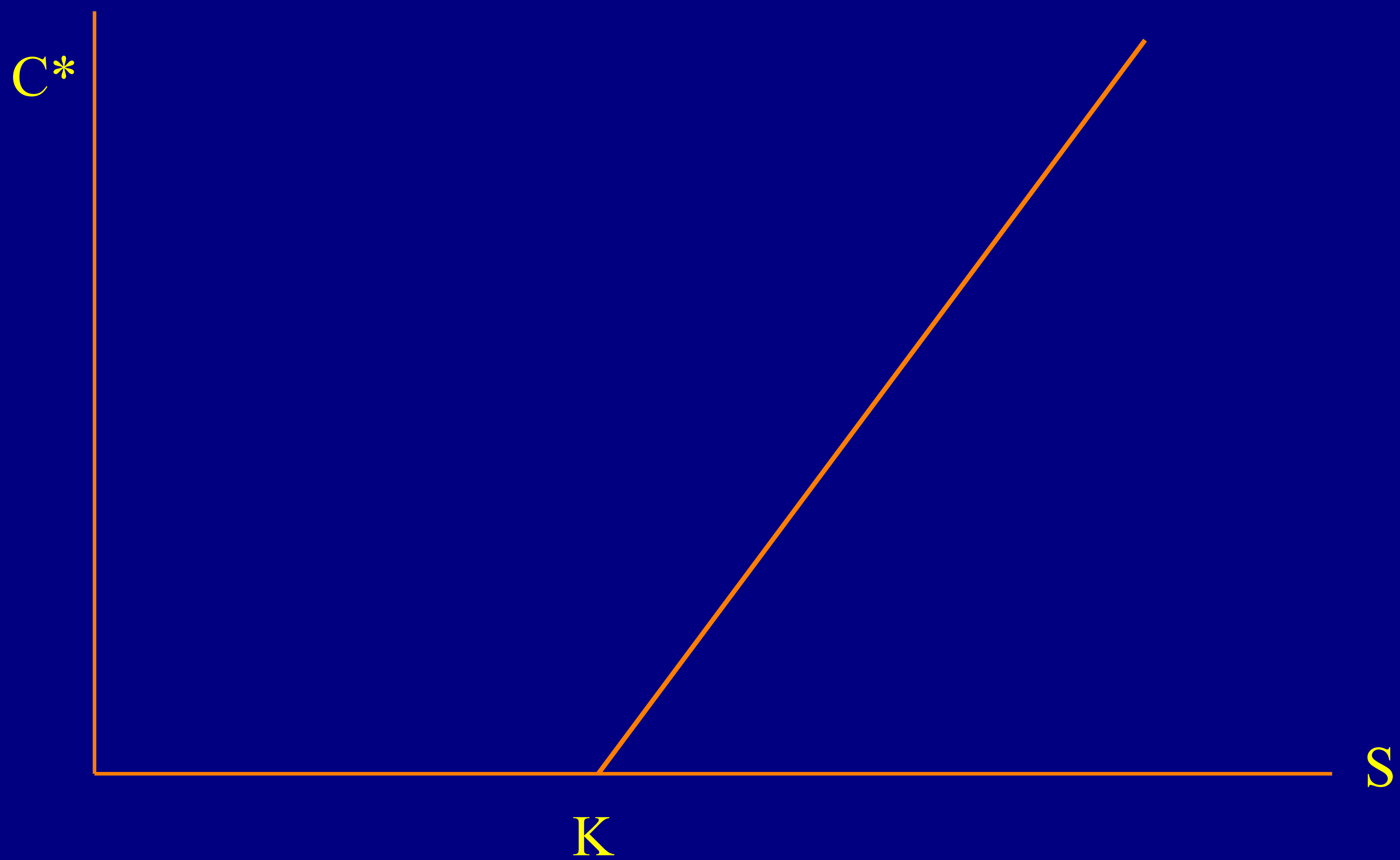
A hedge is risk-free if
Qs / Qc = - ∂C/ ∂S

On Wall Street this is referred to as a delta (Δ) hedge

Comparative Statics of the Black-Scholes Model

$$\overset{+}{}\quad\overset{-}{}\quad\overset{+}{}\quad\overset{?}{}\quad\overset{+}{}$$

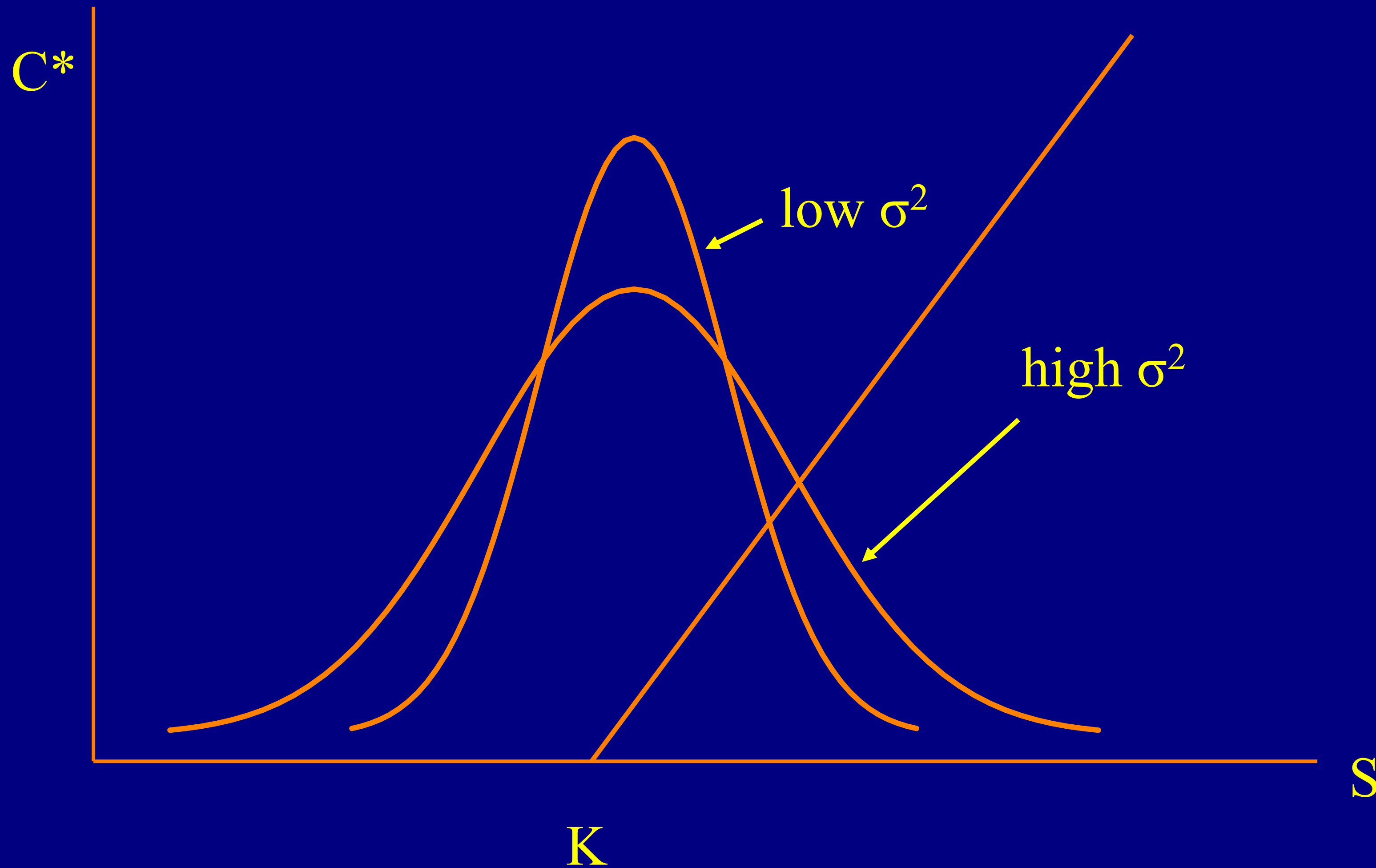
$$C = C(\overset{+}{S}, \overset{-}{K}, \overset{+}{T}, \overset{?}{\sigma^2}, \overset{+}{r}, \text{div})$$



Comparative Statics of the Black-Scholes Model

$$\overset{+}{} \quad \overset{-}{} \quad \overset{+}{} \quad \overset{+}{} \quad \overset{+}{} \quad \overset{-}{}$$

$$C = C(S, K, T, \sigma^2, r, div)$$



Put-Call Parity

With European options, there is a direct relation between put and call options:
Put is equal to a call, minus the stock plus the discounted exercise price:

$$P = C - S + K\,exp(-rT)$$

So the Black-Scholes model (without dividends) can price puts by pricing the call

Valuing Put Options on Dividend Paying Stocks

- It is not true, in general, that a put option is worth more alive than dead
- The optimal exercise strategy for American put options is more complicated than the optimal exercise policy for American call options
- The most common time to exercise an American put option is just after an ex-dividend day, but this is not always the case
- There are likely to be bigger differences between B/S prices and market prices for puts than calls

Estimating Volatility Using Stock Returns

We have already seen that stock volatility changes over time (and is autocorrelated)

- Most people use daily return for a recent period (e.g., a year) to estimate the standard deviation of the stock return σ

- Unusually large(small) estimates of σ usually over-estimate (under-estimate) the option price
  - because of the estimation error

Estimating Volatility Using Option Prices

If you assume the BS model is correct

- You can observe all of the other variables necessary to calculate model prices
- Experiment with different values of σ until you find one that is consistent with the observed option price
- This is called the implied volatility

Using Implied Volatility

If you only have one option, this is not very useful (e.g., in looking for profit opportunities) because you have to assume that the option price is right

- Usually, however, there is a set of options traded with different strike prices for a given maturity, and the implied volatility should be the same for all of these options

- Buy underpriced calls and sell overpriced calls ("buy low and sell high")

  - Since implied volatility is a positive function of the call price, this is a simple rubric

Caveat Emptor

Potential problems with the BS model:

- Stock prices may not follow a lognormal random walk
  - Jumps (like the 10/19/87 market crash)
  - Changing volatility
  - Asymmetric distribution of price drops
- Inside information can be a bigger problem
  - How would you trade if you knew a takeover bid was about to be announced?

Example with Facebook Options

JANUARY 2019 (EXPIRATION: 01/18)							
Calls	Last Sale	Bid	Ask	Puts	Last Sale	Bid	Ask
19 Jan 170.00 (FB1918A170)	26.34	26.20	26.50	19 Jan 170.00 (FB1918M170)	8.45	8.35	8.55
19 Jan 175.00 (FB1918A175)	22.93	22.95	23.25	19 Jan 175.00 (FB1918M175)	10.12	10.05	10.25
19 Jan 180.00 (FB1918A180)	20.00	20.00	20.25	19 Jan 180.00 (FB1918M180)	12.09	12.00	12.25
19 Jan 185.00 (FB1918A185)	17.28	17.20	17.45	19 Jan 185.00 (FB1918M185)	14.40	14.20	14.50

Example with Facebook Options

- I selected a small subset of put and call prices that all had the same maturity (January 18, 2019)

- Exercise prices are close to the current price (S=184.92)
  - X = $170, 175, 180, 185

- All of these options are actively traded
  - Indicated by Volume, Open Interest, and last sale between current bid and ask prices

Example with Facebook Options

- Two different ways to calculate option values

- Excel spreadsheet:
  - options.xlsm

- CBOE web page:
  - http://www.cboe.com/framed/IVolframed.aspx

BLACK-SCHOLES
OPTION PRICING MODEL

This worksheet uses the Black-Scholes option pricing model to calculate European call and put option prices. To use the worksheet, supply the required parameters in the green box (B19 to B23). The call and put prices are displayed in the red box (E20 and E22).

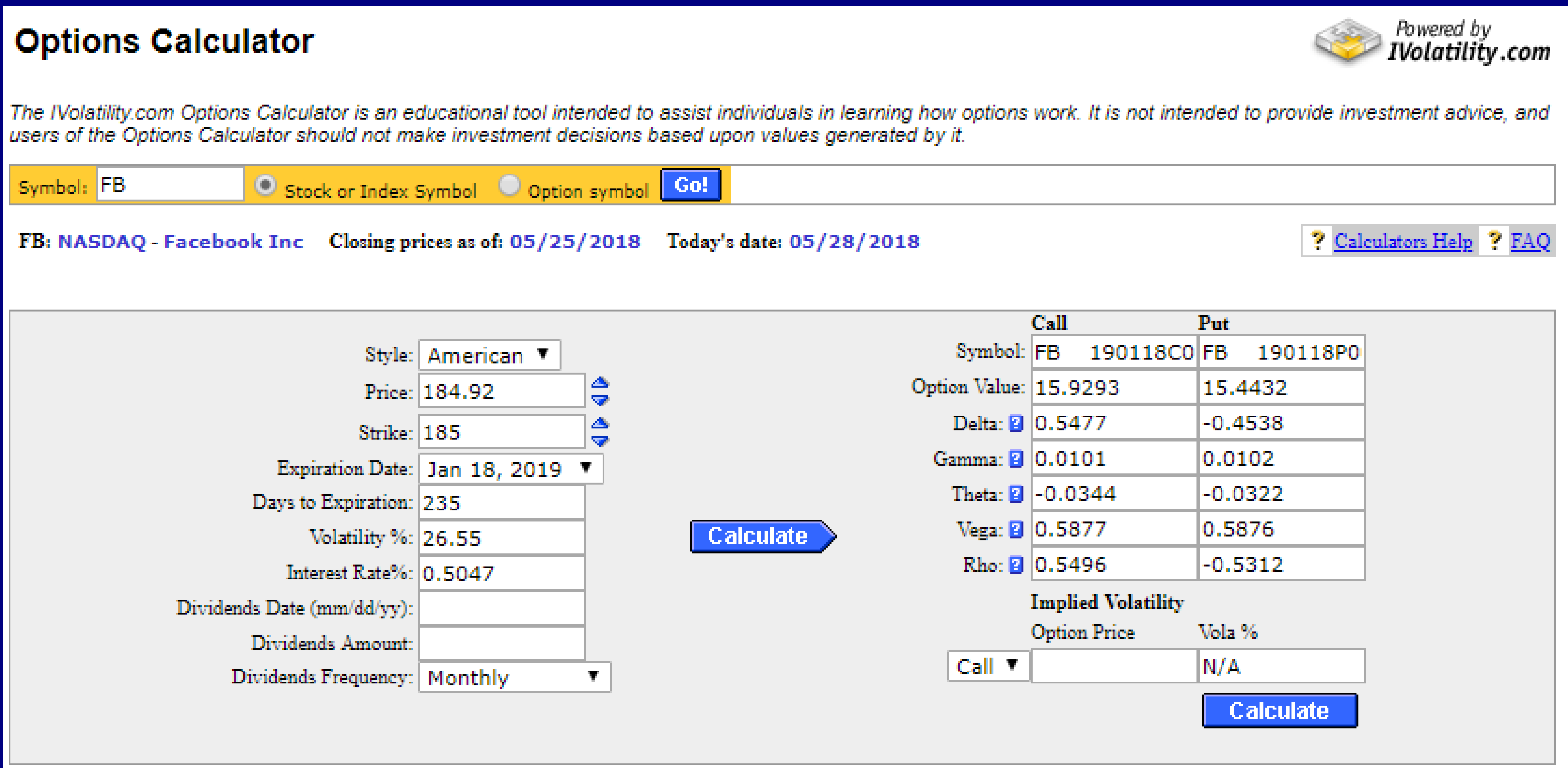
C=S N{d1} - K exp(-rt) N{d2}
where:
S is the current stock price
K is the exercise price for the call and put

t is the time to expiration of the options (e.g., .25 = 91 days out of 365 days in a year)
r is the riskless, continuously compounded, interest rate
(measured in the same units as t)
sigma is the standard deviation of the stock return
(measured in the same units as t)

S	184.920
K	185.000
t	0.644
r	0.005
sigma	0.266
d1	0.1197
d2	-0.0933

Call Price	15.926
Put Price	15.406

Example with Facebook Options: CBOE web page



Example with Facebook Options: Implied Volatility Spreadsheet

BLACK-SCHOLES MODEL
FOR IMPLIED VOLATILITY

This worksheet uses the Black-Scholes option pricing model and Excel "Goal Seek" to calculate the implied volatility from European call option prices. To use the worksheet, supply the required parameters in the green boxes (B20 to B23 and E20 to E21). Then click the "Start" button to initiate goal seek. If the value of "Difference" (in the yellow box, H23) is not close to zero, try another value for "Volatility Guess" (E21). The Implied Volatility answer is in the yellow box (E23).

C=S N{d1} - K exp(-rt) N{d2}
where:
S is the current stock price
K is the exercise price for the call and put
t is the time to expiration of the options (e.g., .25 = 91 days out of 365 days in a year)
r is the riskless, continuously compounded, interest rate
(measured in the same units as t)
sigma is the standard deviation of the stock return
(measured in the same units as t)

Start

S	184.200	Target Call	17.280		
K	185.000	Volatility Guess	0.100		
t	0.644			Call Price	17.280
r	0.005	Implied Volatility	0.295	Difference	0.000
d1	0.1139				
d2	-0.123				

Example with Facebook Options

- Implied volatility is higher for Jan 2019 X=185 call than for the Black-Scholes value because the market price is slightly higher than the model price
- Remember, all calls with the same expiration date should have the same implied volatility

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Data used for these slides can be accessed at:

http:\\schwert.ssb.rochester.edu\brn481\brn481opt.xlsx

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http:\\schwert.ssb.rochester.edu\brn481\options.xlsm

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