. . . About the Term Structure of Interest Rates

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This column addresses the term structure of interest rates. I begin by reviewing the various ways in which the term structure is measured. Then I present the major hypotheses that purport to explain the relationship between interest rates and term to maturity. Finally, I discuss a simple technique for estimating the term structure.

What is the Term Structure of Interest Rates?

The term structure of interest rates, sometimes referred to as the yield curve, isolates the differences in interest rates that correspond solely to differences in term to maturity. As a first approximation, we can measure the term structure by measuring the relationship between the yields to maturity on government debt instruments and their terms to maturity. By focusing on government debt instruments, we control for differences in yield that might arise from credit risk.

The *yield to maturity* of a bond equals the internal rate of return that discounts its cash flows, including the coupon payments and the repayment of principal, back to the bond's current price. This relationship is described by Equation (1):

Eq. 1

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \ldots + \frac{C_n}{(1+y)^n} + \frac{F}{(1+y)^n}$$

where

P = current price,
C_1, C_2, C_n = coupon payments in periods 1 through n,
F = face value,
y = yield to maturity and
n = number of discounting periods.

The yield to maturity does not provide a particularly satisfying yardstick for measuring the term structure of interest rates, for two reasons. First of all, it is an unrealistic measure of a bond's yield because it assumes that all of a bond's cash flows are reinvested at the same rate. This assumption implies that a one-year instrument nine years hence will have the same yield as a 10-year bond today. Obviously, there is no reason to expect interest rates to evolve according to this assumption.

Second, the yield to maturity is deficient because it varies as a function of a bond's coupon rate. If a bond's coupon rate is below its yield to maturity, it will sell at a discount to its face value. Part of its return will thus arise from a capital gain as it gravitates to its face value. Because capital gains receive favorable tax treatment relative to income, bonds that derive their return from income only must offer a higher yield to maturity than discount bonds in order to compete on an after-tax basis. Consequently, bonds with the same term to maturity will have yields to maturity that differ according to the fraction of their return that arises from income versus price change.

To control for the differential tax treatment of coupons and price change, we can measure the term structure of interest rates from the yields on pure discount bonds. These bonds do not pay coupons. Instead, they are initially offered at a discount to their face value, so that their yield is equal to the annualized return resulting from their conversion to face value. The yield on a pure discount bond is referred to as the *spot rate of interest.*

We can think of a bond with predictable cash flows as a portfolio of pure discount bonds. In order to price coupon-bearing bonds using spot interest rates, we assign the yield of a pure discount instrument maturing in six months to the coupon payment six months from now and the yield of a one-year, pure discount instrument to the coupon due one year from now, proceeding in this fashion until we assign yields to all the bond's cash flows. This relationship is shown in Equation (2). For purposes of simplification, Equation (2) assumes that coupon payments occur annually.

Eq. 2

$$P = \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_2)^2} + \ldots + \frac{C_n}{(1+r_n)^n} + \frac{F}{(1+r_n)^n}$$

where

r_1, r_2, r_n = spot rates of interest of pure discount bonds maturing in periods 1 through n.

As long as there is a reasonable supply of pure discount bonds at all relevant maturities, the spot rates of interest should reflect accurately the term structure of interest rates. It might be the case, however, that at particular maturities there is an inadequate supply of pure discount bonds, including coupons that have been stripped from coupon-bearing bonds. In this case, the yields on these bonds might misrepresent the term structure of interest rates. I will address this problem in the final section.

We can also describe the term structure of interest rates by measuring the relationship between forward rates and term to maturity. The *forward rate* is the interest rate that will apply to an instrument commencing at some future date. It can be derived from the spot rates of interest.

Suppose that the spot rate of interest on a one-year instrument is 6.00% and that the spot rate of interest on a two-year instrument is 7.50%. If we were to contract to purchase a one-year instrument one year from now, what rate of interest should we expect for this instrument? The forward rate on a one-year instrument one year hence is determined so that an investor is indifferent between purchasing a two-year instrument today and holding it to maturity or purchasing a one-year instrument today and entering into a forward contract to purchase a one-year instrument one year from now. This equality is shown in Equation (3).

Eq. 3

$$(1 + r_2)^2 = (1 + r_1) \cdot (1 + f_{1,1})$$

where

r_2 = spot rate for two-year instrument,
r_1 = spot rate for one-year instrument and
$f_{1,1}$ = one-year forward rate for one-year instrument.

If we substitute the one and two-year spot rates into Equation (3), we find that the rate on a one-year instrument one year forward equals 9.02%.

Suppose that the market offers a one-year forward rate on a one-year instrument equal to 8.00%. In this case, we would invest in the two-year instrument today because we would be sure to earn a cumulative return of 15.56% ($1.075 \cdot 1.075 - 1$), compared with a cumulative return of 14.48% ($1.06 \cdot 1.08 - 1$) were we to invest in a one-year instrument today and a forward contract to invest in a one-year instrument one year hence. By the same logic, we would choose the one-year instrument and the forward contract if the forward rate were greater than 9.02%. The forward rate is governed by the law of one price, which states that equivalent cash flows must sell for the same price.

In general, we can derive the forward rate for any future date and for instruments of any maturity using Equation (4), provided we can observe instruments with the requisite maturities today.

Eq. 4

$$f_{t,n-t} = [(1 + r_n)^n/(1 + r_t)^t]^{1/(n-t)} - 1$$

where

$f_{t,n-t}$ = t-year forward rate for n − t year instrument,
r_n = spot rate for n-year instrument and
r_t = spot rate for t-year instrument.

Yet another way in which we can represent the term structure of interest rates is to relate *discount factors* to maturity. The discount factor is equal to the reciprocal of one plus the spot rate raised to the maturity of the instrument, as shown in Equation (5).

Eq. 5

$$d(n) = 1/(1 + r_n)^n$$

Table I Discount Factors When Spot Rate of Interest = 8%

Term to Maturity	*Discount Factor*
1	.9259
2	.8573
3	.7938
4	.7350
5	.6806
6	.6302
7	.5835
8	.5403
9	.5002
10	.4632
15	.3152
20	.2145

where

$d(n)$ = discount factor for n periods,
r_n = spot rate of interest for maturity n and
n = maturity of pure discount instrument.

The discount factor must fall between 0 and 1. It approaches 0 as the term to maturity approaches infinity, and it approaches 1 as the term to maturity approaches 0. Consider, for example, a situation in which the spot rates of interest are 8% across all maturities. Table I shows the discount factors corresponding to various maturities.

It is apparent from Table I that the discount factor is a nonlinear function of term to maturity. An increase of five years beginning with a term to maturity of one year reduces the discount factor by nearly 0.3 unit, whereas an increase of five years beginning in year 10 reduces it by less than 0.15 unit and beginning in year 15 by only 0.1 unit. As a percentage of value, however, the discount factor adjusts price proportionately with time. In all cases, an increase in term to maturity of five years reduces the value of the bond by 31.2%, given an 8% spot rate of interest.

What Determines the Term Structure of Interest Rates?

There are three hypotheses that are commonly cited to explain the term structure of interest rates—the expectations hypothesis; the liquidity premium hypothesis; and the segmented market hypothesis, also known as the preferred habitat hypothesis.

The expectations hypothesis holds that the current term structure of interest rates is determined by the consensus forecast of future interest rates. Suppose that the spot interest rate for a one-year instrument is 6% and that the spot rate of interest for a two-year instrument is 7%. According to the expectations hypothesis, this term structure arises from the fact that investors believe that a one-year instrument one year in the future will yield 8.01%, because an investor could achieve the same return by investing in a one-year instrument today and a one-year instrument one year from now as she could achieve by investing in a two-year instrument today. If the investor believes that the one-year rate one year in the future will exceed 8.01%, she will prefer to roll over consecutive one-year instruments as opposed to investing in a two-year instrument today. But if she anticipates that the one-year rate one year ahead will be less than 8.01%, she will opt for the two-year instrument today.

According to the expectations hypothesis, an upward sloping yield curve indicates that investors expect interest rates to rise. A flat yield curve implies that investors expect rates to remain the same. A downward sloping yield curve indicates that investors expect rates to fall.

It is important to distinguish the future spot rates that are implied by the current term structure from the forward rates on contracts available today. Although both rates are calculated in the same way and are therefore equal to each other, the interest rate on a forward contract must obtain in an arbitrage-free world, whereas the implied future spot rate is only a forecast, and not a particularly good one at that.

Based on the term structure in the previous example, the interest rate on a forward contract to purchase a one-year instrument one year from now must equal 8.01%. If the rate were lower, we could sell the forward contract together with a one-year instrument and use the proceeds to purchase a two-year instrument, thereby earning a riskless profit. If the rate on the forward contract were higher, we could reverse these transactions for a riskless profit.

It is not the case, however, that we would necessarily profit by combining purchases and sales today with purchases or sales in the future. Whether or not we profit would depend on the future, not forward, term structure of interest rates.

The distinction between actual forward rates and implied future rates is analogous to the difference between covered interest rate parity and uncovered interest rate parity. Covered interest rate parity is an arbitrage condition that explains the relationship between the spot exchange rate on a currency and its forward exchange rate. The forward rate is set such that an arbitrager cannot profit by borrowing in a low-interest-rate country, converting to the currency of a high-interest-rate country, lending at the higher interest rate and selling a forward contract to hedge away the currency risk. The cost of the hedge will precisely offset the interest rate advantage.

Uncovered interest rate parity posits that, on average, we cannot profit by borrowing in a low-interest-rate country, converting to the currency of a high-interest-rate country, and lending in that country without hedging away the currency risk. In effect, uncovered interest rate parity is nothing more than a statement that the forward rate is an unbiased estimate of the future spot rate. "Unbiased" does not mean that the forward rate is an accurate forecast of the future spot rate. It merely suggests that it does not systematically over or underestimate the future spot rate.

It is implausible that the expectations hypothesis fully accounts for the term structure of interest rates. As mentioned earlier, when the term structure of interest rates slopes upward, according to the expectations hypothesis, investors expect interest rates to rise. Historically, the term structure has had an upward slope about 80% of the time. It seems unlikely that investors have expected interest rates to rise with that degree of frequency.

The expectations hypothesis is implausible for another reason. In order for it to be true, investors must believe that all bonds will generate the riskless return. Suppose that the spot rates of interest on a one-year instrument, a four-year instrument and a five-year instrument equal 6%, 7.5% and 8%, respectively. The expectations hypothesis implies that a four-year instrument one year in the future will have a rate of 8.51%. Based on today's term structure, we could purchase a pure discount bond with a face value of $1,000 maturing in five years for $680.58. If the implied future rate of 8.51% on a four-year discount bond is realized one year from now, we could then sell our four-year bond for $721.42, thereby earning a return of precisely 6.0%, which equals the riskless return on a one-year instrument.

Table II shows the implied term structure one year from now, given the present term structure along with the total return one would achieve during the ensuing year by purchasing discount bonds of various maturities should the implied future term structure materialize.

The implicit forecast that all bonds will yield the riskless return challenges credulity. It suggests that investors are indifferent to risk. Historically, the returns of long-term bonds have been

Table II Total Return Implied By Forward Rates

Term To Maturity	Current Spot Rates	Implied Future Rates One Year Forward	Implied Total Return
1 Year	6.00%	7.00%	6.00%
2 Years	6.50%	7.50%	6.00%
3 Years	7.00%	8.00%	6.00%
4 Years	7.50%	8.51%	6.00%
5 Years	8.00%		6.00%

higher on average and significantly more volatile than the returns on short-term instruments. This evidence suggests that investors demand and receive a premium in exchange for the higher volatility of long-term bonds.

The higher historical returns of long-term bonds relative to shorter-term instruments lend credence to an alternative explanation of the term structure—the *liquidity premium hypothesis*. This hypothesis holds that investors are not indifferent to risk. They recognize that a bond's price is more sensitive to changes in interest rates, the longer its maturity, and they demand compensation for bearing this interest rate risk. Thus bonds with longer maturities typically offer a premium in their yields relative to shorter-term instruments in order to induce investors to take on additional risk. The extent of the premium increases with term to maturity but at a decreasing rate, for two reasons. Duration, a measure of a bond's price sensitivity to interest rate changes, increases at a decreasing rate with term to maturity.[1] Moreover, long-term interest rates are typically less volatile than short-term interest rates.

The notion that yields on longer-term instruments reflect a liquidity premium is consistent with the observation that the yield curve usually has an upward slope. Even when investors anticipate that interest rates will remain the same or decline slightly, a liquidity premium could still cause long-term rates to exceed short-term rates.

A third explanation of the term structure of interest rates is the *segmented market hypothesis*, which holds that groups of investors regularly prefer bonds within particular maturity ranges in order to hedge their liabilities or to comply with regulatory requirements. Life insurance companies, for example, have historically preferred to purchase long-term bonds, whereas commercial banks have favored shorter-term instruments. To the extent the demand of one group of investors increases relative to the demand of the other group, yields within the maturity range where relative demand has risen will fall relative to the yields within the maturity range where there is slack in demand.

Term Structure Estimation

If we were to trace a line through the yields on pure discount government bonds as they relate to maturity, it is unlikely that this line would form a smooth curve. Some of the observations would likely rise abruptly and some would likely fall abruptly. These apparent jumps might reflect the fact that some of the instruments have not traded recently; thus the observations are not contemporaneous with each other. Moreover, it may be the case that we have no observations for some maturities. In an effort to overcome these limitations, financial analysts have developed methods for estimating a smooth curve to represent the term structure.

One such method is called *spline smoothing*. This approach assumes that the discount factors corresponding to the spot interest rates are a cubic function of time to maturity, as shown in Equation (6):

Eq. 6

$$d(n) = a + b \cdot n + c \cdot n^2 + d \cdot n^3$$

where

$d(n)$ = discount factor for maturity n and

n = term to maturity.

We estimate the coefficients of Equation (6) by regressing the observed discount factors on three independent variables—term to maturity, its value squared and its value cubed. We then convert the estimated discount factor to its corresponding yield.

Table III shows a hypothetical observed term structure along with an estimated term structure based on the cubic spline method. The fitted regression equation from the observed term structure is as follows:

$$\text{Discount Factor} = 1.01508 - 0.06206 \cdot n - 0.00017 \cdot n^2 + 0.000087 \cdot n^3.$$

Table III Cubic Spline Estimated Term Structure

Maturity	Observed Spot Rate	Observed Discount Factor	Estimated Discount Factor	Estimated Spot Rate
1	5.03%	.9521	.9529	4.94%
2	5.89%	.8918	.8909	5.94%
3	6.47%	.8285	.8297	6.42%
4	6.57%	.7753	.7696	6.77%
5	7.20%	.7064	.7112	7.05%
6	7.35%	.6534	.6552	7.30%
7	7.55%	.6008	.6019	7.52%
8	7.60%	.5565	.5519	7.71%
9	7.89%	.5049	.5057	7.87%
10	8.00%	.4632	.4639	7.98%

These estimated yields allow us to price bonds for which we do not have reliable observations. Furthermore, we may believe that the values along the smoothed curve represent yields toward which the observed yields will converge, thereby suggesting trading opportunities.

I have attempted to provide a broad overview of the term structure of interest rates. This topic is one of the most widely researched areas of finance; the interested reader will have no trouble pursuing more detailed analyses.[2]

Footnotes

1. *For a discussion of duration, see M. Kritzman, "What Practitioners Need to Know About Duration and Convexity,"* Financial Analysts Journal, *November/December 1992.*
2. *For example, see S. Brown and P. Dybvig, "The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates,"* Journal of Finance, *July 1986; J. Cox, J. Ingersoll and S. Ross, "A Reexamination of Traditional Hypotheses About the Term Structure of Interest Rates,"* Journal of Finance, *September 1981; H. G. Fong and O. Vasicek, "Term Structure Modeling,"* Journal of Finance, *July 1982; and C. Nelson and A. Siegel, "Parsimonious Modeling of Yield Curves,"* Journal of Business *60 (1987), no. 4.*