. . . About Option Replication

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A financial option gives the owner the right to buy (in the case of a call option) or to sell (in the case of a put option) an asset at a specified price. This right, in the case of an American option, can be exercised at any time over a specified period. (A European option can be exercised only at a specific date.) Exchange-traded options are available on a large variety of individual securities, as well as on stock indexes, dollar-denominated foreign currencies and futures contracts.

Despite the wide availability of exchange-traded options, the demand for option-like payoffs vastly exceeds the supply. A large market in privately negotiated options and in option-replication strategies has emerged. This generates option-like results for assets or portfolios on which exchange-traded options are not available or for terms different from the typical terms of exchange-traded options. This column demonstrates how we can generate an option-like payoff by shifting a fund between two assets.

Hedging Currency Risk

Suppose a corporation based in London expects to receive 75 million deutschemarks one year from today. The CFO is concerned that the mark might depreciate relative to the pound during this period. Let us assume that the spot exchange rate currently equals three marks per pound, so that the receivable is worth 25 million pounds today. What are the CFO's alternatives for hedging this risk?

One approach is to sell 75 million marks forward. Under this strategy, if the mark depreciates, as feared by the CFO, the loss that will occur when the 75 million marks are converted into pounds will be offset exactly by the gain on the forward contract. That is the good news. The bad news is that any gain, should the mark instead *appreciate* relative to the pound, will be offset by an equivalent loss on the forward contract.

Ideally, the CFO would like to protect the firm's receipts against a possible decline in the mark relative to the pound and to profit in the event the mark appreciates. An option to exchange 75 million marks for 25 million pounds one year from now would achieve these contingent results. If the mark depreciates, the CFO collects the 75 million marks and, under the terms of the option contract, exchanges them for 25 million pounds regardless of the prevailing exchange rate. Alternatively, if the mark appreciates, the option expires worthless, but the CFO exchanges the marks for pounds at the higher exchange rate.

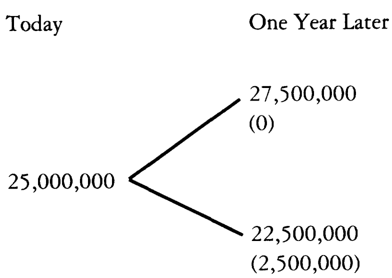
Of course, this privilege is not without cost. The CFO must pay a premium for the option to exchange marks for pounds at today's exchange rate. She will suffer a net loss if the mark depreciates or fails to appreciate sufficiently to offset the option premium. Her maximum loss is limited to the amount of the option premium (assuming the exercise price is based on the current exchange rate), while her potential gain is unlimited. Therefore, let us suppose that she chooses to pursue the option strategy.

How should she proceed? There are no exchange-traded options to convert marks to pounds one year forward, nor can such options be constructed by combining exchange-traded options to exchange pounds for dollars and dollars for marks. But many financial institutions, including large brokerage firms and banks, offer privately negotiated options to meet specific customer requirements. These institutions are motivated by the fact that they can usually hedge the risk exposure incurred by writing the option at a lower cost than the premium they charge.

Option Replication Using the Binomial Model

Under perfect market conditions, one can replicate the contingent payoff of an option by shifting funds between a riskless asset (in our example, the one-year Treasury bill in the United Kingdom) and a risky asset (the mark). In order to demonstrate this correspondence, let us begin by making the simplifying assumption that the mark, which we assumed earlier can be exchanged for 0.3333 pounds today, will either increase 10 per cent or decrease 10 per cent one year hence, and that these outcomes are equally probable. Moreover, in order to focus the discussion on the essence of option replication, we will make the convenient but obviously false assumption that the riskless rate of interest is 6 per cent in the United Kingdom and 0 per cent in Germany. Thus the only possible values for the receivable of 75 million marks, which today can be exchanged for 25 million pounds, are 27.5 million pounds (mark appreciates 10 per cent) and 22.5 million pounds (mark depreciates 10 per cent), as Figure A shows.

Figure A Binomial Model



We can easily determine the value of an option to exchange 75 million marks for 25 million pounds one year hence. The option will be worth nothing if the mark appreciates to 0.3667, because at this value 75 million marks can be exchanged for 27.5 million pounds. If the mark falls in value to 0.3000, however, the option will be worth 2.5 million pounds—the difference between the 25 million pounds for which the marks could be exchanged under the terms of the option contract and 22.5 million pounds, which is the pound-equivalent of 75 million marks at an exchange rate of 0.3000. These contingent values are shown in parentheses in Figure A.

Given the obvious values for an option to exchange marks for pounds one year from now, we can construct two equations that, when solved simultaneously, reveal how we can hedge the option by combining exposure to the mark and a riskless asset:

$$27{,}500{,}000 \cdot M + 1.06 \cdot P = 0,$$

$$22{,}500{,}000 \cdot M + 1.06 \cdot P$$

$$= 2{,}500{,}000.$$

Here M represents exposure to the mark, while P represents exposure to the U.K. Treasury bill, both from the perspective of our British CFO. Recall our earlier assumptions that the riskless return is 6 per cent in the U.K. and 0 per cent in Germany. By solving these equations simultaneously, we find that M equals −0.5000 and P equals 12,971,698. This indicates that we can hedge an option to exchange 75 million marks for 25 million pounds one year from now by selling marks short in an amount equal to 12.5 million pounds and investing 12,971,698 pounds at 6 per cent, the riskless rate of interest in the U.K. Furthermore, the "fair value" of this option today must equal the sum of our short position in the mark and our long position in the U.K. Treasury bill, which equals 471,698 pounds.

To validate the equivalence between this hedging strategy and an option to exchange 75 million marks for 25 million pounds one year forward, let us consider the payoff to the institution that writes the option and then hedges it as we just described, assuming it charges the aforementioned fair value of 471,698 pounds as the premium. If the mark rises 10 per cent, the option will expire worthless. At the same time, the short position in the mark will produce a loss of 1,250,000 pounds, while the long position in the Treasury bill will yield a gain of 778,302 pounds for a net loss to the hedging strategy of 471,698 pounds. This loss exactly offsets the premium received for writing the option.

If the mark declines 10 per cent, the option will be worth 2,500,000 pounds. Thus the net loss for writing the option will equal 2,028,302 pounds. The short position in the mark, however, will yield a profit of 1,250,000 pounds, which when combined with the yield from the investment in the Treasury bill of 778,302 pounds totals 2,028,302 pounds. This exactly offsets the net loss from writing the option. Table I summarizes these results.

From the payoffs described in this example, it easy to see that the institution writing the option can earn a riskless profit as long as it charges a premium in excess of 471,698 pounds. These results, however, depend on the simplifying assumption that the mark can only increase or decrease by 10 per cent in the course of a year, and that both outcomes are equally likely. This assumption is, of course, false. The exchange rate between the pound and the mark changes continually throughout the day, every business day. The basic methodology described above can, however, be expanded to accommodate real-world conditions. Expanding the binomial tree in Figure A to include small changes in the value of the mark over many short time intervals, based on a realistic estimate of the mark's volatility, would allow us to derive a more precise estimate of the option's fair value today and as the year unfolds. In order to hedge this option, we would need to revise the exposure to the mark and the Treasury bill continually as the mark's value changes and as time passes.

Table I Option Payoffs

Mark Rises 10 Per Cent:	
Put Value:	0
Premium Received:	471,698
Net Return from Writing Option:	471,698
Mark Payoff (−12,500,000 · 0.1):	−1,250,000
Treasury Bill Payoff (12,971,698 · 0.06):	778,302
Net Return from Hedging Strategy:	−471,698
Mark Falls 10 Per Cent:	
Put Value (25,000,000–22,500,000):	2,500,000
Premium Received:	471,698
Net Return from Writing Option:	−2,028,302
Mark Payoff (−12,500,000 · −0.1):	1,250,000
Treasury Bill Payoff (12,971,698 · 0.06):	778,302
Net Return from Hedging Strategy:	2,028,302

Option Replication Using the Continuous-Time Black-Scholes Model

As the discrete time intervals in the binomial tree become smaller, approaching zero, the results of the binomial model resemble the results that would be obtained by using a variation of the continuous-time Black-Scholes model.

Based on the insights of the Black-Scholes model, we can determine the value of an option to exchange marks for pounds from a British investor's perspective by using the following formula:

$$Xe^{-r_d T} \cdot N(-D + \sigma\sqrt{T})$$

$$- Me^{-r_f T} \cdot N(-D)$$

where

- M = value of mark denominated in pounds,
- X = strike price,
- T = time remaining until expiration,
- r_d = instantaneous riskless rate of interest in U.K. $\ln(1 + r)$,
- r_f = instantaneous riskless rate of interest in Germany $\ln(1 + r)$,
- D = $[\ln(M/X) + (r_d - r_f + \sigma/2) \cdot T]/(\sigma\sqrt{T})$,
- $\ln(\ )$ = natural log,
- σ = standard deviation of the return on the mark and
- $N(\ )$ = cumulative normal distribution function.

This formula allows us to relax our earlier assumption that Germany's riskless interest rate is 0 per cent. If the foreign riskless rate were 0 per cent, the formula would converge exactly to the Black-Scholes formula for a non-dividend-paying stock. Within the context of a currency option, the interest payment can be thought of as a dividend on the currency so that the formula for valuing a currency option is analogous to the formula used to value a European option on a dividend-paying stock.

In order to illustrate the application of this formula to hedge an option to exchange marks for pounds, let us retain our earlier assumptions and make the additional assumptions that Germany's riskless rate equals 5 per cent and the mark's standard deviation equals 10 per cent. These assumptions are shown below:

M = 25,000,000 pounds,
X = 25,000,000 pounds,
T = one year,
$r = \ln(1.06) = 5.83$ per cent,
$r = \ln(1.05) = 4.88$ per cent and
$\sigma = 10$ per cent.

By substituting these values into the above formula, we discover that the currency option's fair value equals 836,918 pounds. The hedge ratio equals −0.421. This value determines the exposure to the mark, which, when combined with investment in the Treasury bill, hedges the option. To hedge an option to exchange 75 million marks for 25 million pounds, we must thus start out by selling marks short in an amount equivalent to 10,534,272 pounds. By netting this short position out of our original exposure of 25 million pounds and adding to it the value of the option, we discover the amount to invest in Treasury bills, which equals 15,302,645 pounds.

The exposures derived above represent the starting positions to hedge the option. As the exchange rate between the mark and the pound changes, and as time passes, we must repeat the exercise to determine the current exposures; this explains the term "dynamic hedging." As long as the mark's exchange rate does not change so abruptly that we cannot execute the requisite trades and as long as transaction costs are not a factor, this dynamic hedging strategy should effectively hedge an option to exchange 75 million marks for 25 million pounds one year from now.

Because the institution that writes the option is contractually obligated to deliver 25 million pounds in exchange for 75 million marks should the mark's value decline, it bears the risk that its hedging strategy may not perform as expected. As compensation for bearing this risk, and to cover transaction costs, the institution will charge a higher premium than the theoretical premium implied by the option pricing formula. If the spread between the actual premium and the theoretically fair premium is judged to be unreasonably wide, the customer always has the option to self-insure by implementing the dynamic hedging strategy internally.

Guest Speaker *concluded from p. 10.*

inflation is only cyclically depressed. At least moderate inflation has been and in my judgment will be a continuing characteristic of the U.S. economy.

Footnotes

1. *It is interesting that the Bundesbank, perhaps the world's most vigorously independent central bank, is feeling analogous pressures to reorganize to reflect the changing structure of Germany and the demands for more rapid economic growth.*
2. *In roughly chronological order, Penn Central, Franklin National, New York City, Chrysler, Lockheed, Continental Illinois, Farm Credit, Federal Savings and Loan Insurance Corporation, Bank of New England, Federal Deposit Insurance Corporation, etc.*
3. *It is an even higher-risk bet for endowments of labor-intensive foundations and educational institutions, as well as for taxable investors whose taxes are based on nominal rather than real gains.*