Sample Answer

I modeled the conditional heteroskedasticity of stock returns and, separately, the risk premium using four different general models: a second order ARCH model, a GARCH (1,2) model, a threshold ARCH (1,2) model and an exponential GARCH (1,2) model. The mean equation for each model was specified to include only a constant:

$$r_t = c + \varepsilon_t$$

where r_t is the stock return (or risk premium) for month t and ε_t is an error term with conditional variance as specified in the particular model (to be discussed below). Each specification was estimated with and without the inclusion of the NBER recession dummy variable (REC) as part of the conditional variance equation. This dummy proxies as a type of "regime-switching" parameter, although the definition of the regimes is determined exogenously (and ex-post). Essentially, the dummy is used to test whether stock volatility tends to be higher during recessionary periods. Each model was estimated using numerical maximum likelihood estimation assuming a normal conditional distribution for the errors. Discussion of the models and the results of the estimation procedures follow, as do comparisons between the different models.

ARCH (2) Model

The ARCH (2) specification models the conditional variance of the error term from the mean equation (σ_t^2) as follows:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \delta\varepsilon_{t-2}^2 + \rho REC_t$$

So this period's conditional variance depends upon the square of the error from the mean equation from last period and the period before, as well as the dummy variable representing NBER recessions (REC) when included. This specification allows the magnitude of recent shocks to affect the conditional variance of this period's shock.

Table 1

ARCH (2) model coefficient estimates (standard errors in parentheses) for conditional variance of stock returns. Estimates are maximum likelihood estimates of the parameters of the following model for the conditional variance:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \delta\varepsilon_{t-2}^2 + \rho REC_t$$

	ω	α	δ	ρ
A. Full period (1885:01 - 1998:09)				
Without REC	0.001534 (0.000059)	0.099159 (0.018664)	0.303489 (0.026905)	
With REC	0.001357 (0.000052)	0.078730 (0.016716)	0.252298 (0.025123)	0.001137 (0.000166)
B. Middle period (1926:01 - 1998:09)				
Without REC	0.001573 (0.000073)	0.132020 (0.028876)	0.349851 (0.035056)	
With REC	0.001391 (0.000060)	0.082819 (0.023924)	0.237950 (0.027038)	0.002759 (0.000398)
C. Later period (1953:01 - 1998:09)				
Without REC	0.001465 (0.000091)	0.056745 (0.046136)	0.095904 (0.056576)	
With REC	0.001414 (0.000081)	0.043059 (0.044778)	0.004579 (0.048605)	0.001425 (0.000505)

Table 1 presents the results from estimating this model for the three specified sample periods. Models which included the NBER dummy variable are marked "With REC." For both the full period and the middle period, both ARCH terms enter the equation significantly, as does REC when included. Positive coefficients indicate that there is persistence in volatility shocks – large shocks are often followed by large shocks. The coefficient estimates are also similar across samples. However, for the later period, only REC is significantly different from zero. This model seems to have little power to characterize conditional volatility for the 1953 to 1998 period.

GARCH (1,2) Model

The GARCH (1,2) specification models conditional variance as follows:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \delta\varepsilon_{t-2}^2 + \beta\sigma_{t-1}^2 + \rho REC_t$$

This specification is identical to that of the ARCH (2) above except for the addition of last period's conditional variance as an explanatory variable. This allows prior conditional variances to affect current conditional variances in addition to prior (realized) shocks. Table 2 presents the results from the estimation of this model for all three sample

Table 2

GARCH (1,2) model coefficient estimates (standard errors in parentheses) for conditional variance of stock returns. Estimates are maximum likelihood estimates of the parameters of the following model for the conditional variance:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \delta\varepsilon_{t-2}^2 + \beta\sigma_{t-1}^2 + \rho REC_t$$

	ω	α	δ	β	ρ
A. Full period (1885:01 - 1998:09)					
Without REC	0.000105	0.038421	0.100951	0.821858	
	(0.000023)	(0.025722)	(0.031249)	(0.019770)	
With REC	0.000092	0.032502	0.102146	0.819327	0.000108
	(0.000023)	(0.024329)	(0.030594)	(0.020120)	(0.000038)
B. Middle period (1926:01 - 1998:09)					
Without REC	0.000074	0.062646	0.084254	0.834590	
	(0.000024)	(0.037746)	(0.044122)	(0.021715)	
With REC	0.000074	0.057146	0.080988	0.836365	0.000077
	(0.000023)	(0.036692)	(0.043158)	(0.022876)	(0.000076)
C. Later period (1953:01 - 1998:09)					
Without REC	0.000105	0.037020	0.081266	0.832232	
	(0.000043)	(0.039414)	(0.053253)	(0.042475)	
With REC	0.000112	0.033830	0.079926	0.827316	0.000055
	(0.000046)	(0.038859)	(0.051959)	(0.045507)	(0.000083)

periods. For all sample periods, the first ARCH term is not generally significant, while the GARCH term is very significant. The second ARCH term is highly significant for the whole period, but only marginally so over the subperiods. The NBER recession dummy variable is significant only for the full period. Interestingly, estimation over the later sample period is fairly similar to the longer middle period, in contrast to the ARCH model. Overall, comparing this specification to the ARCH specification, it would appear that the inclusion of the GARCH term is very important. Also note that the sum of the coefficient estimates on the ARCH and GARCH terms is close to unity in each case, indicating significant persistence in volatility ~~shocks.~~

Threshold ARCH (TARCH (1,2)) Model

The TARCH (1,2) specification models conditional variance as follows:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 d_{t-1} + \delta\varepsilon_{t-2}^2 + \beta\sigma_{t-1}^2 + \rho REC_t$$

where $d_t = 1$ if $e_t < 0$ and $d_t = 0$ otherwise. This specification is the same as the GARCH (1,2) specification except that it allows for positive and negative shocks to have asymmetric affects on future conditional volatility. Positive shocks have effect α while negative shocks have effect $\alpha + \gamma$. Table 3 presents results from the estimation of this

Table 3

TARCH (1,2) model coefficient estimates (standard errors in parentheses) for conditional variance of stock returns. Estimates are maximum likelihood estimates of the parameters of the following model for the conditional variance:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 d_{t-1} + \delta\varepsilon_{t-2}^2 + \beta\sigma_{t-1}^2 + \rho REC_t$$

	ω	α	γ	δ	β	ρ
A. Full period (1885:01 - 1998:09)						
Without REC	0.000105	-0.040772	0.112812	0.117434	0.826415	
	(0.000022)	(0.019943)	(0.025546)	(0.026659)	(0.019740)	
With REC	0.000094	-0.042446	0.103448	0.125493	0.821871	0.000079
	(0.000022)	(0.019329)	(0.027115)	(0.026880)	(0.019938)	(0.000039)
B. Middle period (1926:01 - 1998:09)						
Without REC	0.000090	-0.040071	0.128220	0.130542	0.821030	
	(0.000024)	(0.028375)	(0.037456)	(0.036607)	(0.021813)	
With REC	0.000087	-0.040596	0.124755	0.129103	0.822290	0.000047
	(0.000023)	(0.028027)	(0.038366)	(0.036553)	(0.022551)	(0.000079)
C. Later period (1953:01 - 1998:09)						
Without REC	0.000182	-0.131810	0.209143	0.211428	0.739144	
	(0.000051)	(0.017045)	(0.055916)	(0.043043)	(0.039095)	
With REC	0.000159	-0.134315	0.202496	0.229458	0.745430	0.000037
	(0.000048)	(0.017532)	(0.054536)	(0.044344)	(0.035222)	(0.000112)

model for all three sample periods. Interestingly, α, the effect of positive shocks, is negative and significant in all equations, while γ, the additive effect for negative shocks, is positive and significant in all equations, as is the sum of the two. This indicates that the effects of positive and negative shocks are highly asymmetric. Again, however, the GARCH term has the largest and most significant coefficient in each equation, and the NBER recession dummy is significant only for the full sample period. The later period is again unusual, in that the ARCH terms are more significant and the GARCH term significantly smaller than in the longer sample periods. Also, the sum of the ARCH and GARCH coefficients in each equation is near unity, implying significant persistence in volatility ~~shocks.~~

Exponential GARCH (EGARCH (1,2)) Model

The EGARCH (1,2) specification models conditional variance as follows:

$$\ln\sigma_t^2 = \omega + \alpha\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma\frac{\varepsilon_{t-1}}{\sigma_{t-1}} + \delta\left|\frac{\varepsilon_{t-2}}{\sigma_{t-2}}\right| + \lambda\frac{\varepsilon_{t-2}}{\sigma_{t-2}} + \beta\ln\sigma_{t-1}^2 + \rho REC_t$$

This specification models the log of the conditional variance, implying exponential rather that quadratic effects. This model also assures positive variance estimates. In this specification, γ and λ test for asymmetric effects. If they are less than zero, then negative shocks have larger effects on future conditional variance. Table 4 presents results from

Table 4

EGARCH (1,2) model coefficient estimates (standard errors in parentheses) for conditional variance of stock returns. Estimates are maximum likelihood estimates of the parameters of the following model for the conditional variance:

$$\ln\sigma_t^2 = \omega + \alpha\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma\frac{\varepsilon_{t-1}}{\sigma_{t-1}} + \delta\left|\frac{\varepsilon_{t-2}}{\sigma_{t-2}}\right| + \lambda\frac{\varepsilon_{t-2}}{\sigma_{t-2}} + \beta\ln\sigma_{t-1}^2 + \rho REC_t$$

	ω	α	γ	δ	λ	β	ρ
A. Full period (1885:01 - 1998:09)							
Without REC	-0.360860	0.000167	-0.209403	0.229923	0.143024	0.970536	
	(0.048836)	(0.050631)	(0.035140)	(0.055290)	(0.030073)	(0.006404)	
With REC	-0.402995	-0.042834	-0.232722	0.259480	0.189024	0.964894	0.060537
	(0.048976)	(0.053572)	(0.036195)	(0.056343)	(0.032312)	(0.006465)	(0.015534)
B. Middle period (1926:01 - 1998:09)							
Without REC	-0.309484	-0.015481	-0.314416	0.271191	0.259718	0.981722	
	(0.047437)	(0.065695)	(0.045253)	(0.070041)	(0.036264)	(0.006041)	
With REC	-0.357740	-0.052816	-0.331052	0.262402	0.298241	0.970957	0.081151
	(0.048650)	(0.072060)	(0.046648)	(0.074326)	(0.040949)	(0.007195)	(0.023906)
C. Later period (1953:01 - 1998:09)							
Without REC	-0.720720	-0.178035	-0.442456	0.418763	0.372115	0.917895	
	(0.267289)	(0.108199)	(0.058403)	(0.120812)	(0.053029)	(0.037490)	
With REC	-0.802251	-0.187842	-0.443590	0.412564	0.379127	0.904653	0.049897
	(0.295982)	(0.105278)	(0.056626)	(0.117946)	(0.052299)	(0.042457)	(0.053121)

the estimation of this model for all three sample periods. Here, γ is always negative and significant, indicating significant asymmetry (negative shocks increase next period's conditional volatility more than positive shocks), while λ is always significantly positive, indicating an opposite result for the second lag. As with the above models, the second ARCH term always enters more significantly than the first, and the GARCH term is the largest and most significant variable in each equation. REC enters significantly in the full and middle periods. The later period results, again, in more significant ARCH coefficients and smaller GARCH coefficients. The most recent period appears, by all accounts, to be quite unusual.

Comparison of Different Models

In order to determine which models fit the data best, I regress the squares of the residuals from the mean equations for each model on the conditional variance series produced by the variance equation. The equation takes the form:

$$\hat{e}_t^2 = \alpha + \beta\hat{\sigma}_t^2 + \upsilon_t$$

Table 5 presents the results of this analysis for all 24 of the models discussed above. If the forecasts of conditional variance are unbiased, then $\alpha = 0$ and $\beta = 1$. For the full sample period, only the ARCH (2) model without the REC dummy has parameter estimates that are more than one standard error away from these hypothesized values. However, for the middle period, the EGARCH (1,2) model also performs very poorly as measured by this test. For the later period, the GARCH (1,2) and TARCH (1,2) models seem to be severely biased, while the EGARCH (1,2) model performs the best (with parameters closest to these hypothesized values).

Table 5

Comparison of within-sample predictive power of models for the conditional variance of stock returns

OLS estimation results of the following equation (White (1980) heteroskedasticity-consistend standard errors in parentheses):

$$\hat{e}_t^2 = \alpha + \beta\hat{\sigma}_t^2 + \upsilon_t$$

A. Full period (1885:01 - 1998:09)

	α	β	R^2
ARCH (2)			
Without REC	0.000859 (0.000530)	0.675232 (0.247188)	0.053750
With REC	0.000517 (0.000594)	0.824197 (0.280048)	0.061009
GARCH (1,2)			
Without REC	0.000360 (0.000485)	0.857555 (0.241217)	0.118433
With REC	0.000293 (0.000497)	0.887108 (0.247460)	0.120542
TARCH (1,2)			
Without REC	-0.000010 (0.000543)	1.004682 (0.266018)	0.138248
With REC	0.000036 (0.000531)	0.983054 (0.260438)	0.135734
EGARCH (1,2)			
Without REC	-0.000162 (0.000558)	1.091950 (0.278749)	0.132442
With REC	0.000100 (0.000534)	0.971067 (0.262370)	0.132862

B. Middle period (1926:01 - 1998:09)

	α	β	R^2
ARCH (2)			
Without REC	0.001202 (0.000553)	0.599546 (0.230588)	0.059476
With REC	0.000476 (0.000652)	0.874892 (0.284024)	0.077974
GARCH (1,2)			
Without REC	0.000555 (0.000488)	0.792714 (0.226204)	0.129202
With REC	0.000467 (0.000506)	0.831313 (0.235875)	0.131743
TARCH (1,2)			
Without REC	0.000230 (0.000530)	0.901846 (0.243542)	0.144684
With REC	0.000203 (0.000535)	0.913390 (0.246173)	0.145806
EGARCH (1,2)			
Without REC	0.000776 (0.000426)	0.723266 (0.199234)	0.110551
With REC	0.000760 (0.000463)	0.721560 (0.209287)	0.129312

C. Later period (1953:01 - 1998:09)

	α	β	R^2
ARCH (2)			
Without REC	0.000080 (0.000811)	0.954047 (0.473527)	0.011628
With REC	-0.000185 (0.000574)	1.111872 (0.339881)	0.031005
GARCH (1,2)			
Without REC	0.000812 (0.000316)	0.508399 (0.185229)	0.016563
With REC	0.000749 (0.000335)	0.545114 (0.195930)	0.018376
TARCH (1,2)			
Without REC	0.000916 (0.000292)	0.436509 (0.159238)	0.027976
With REC	0.000945 (0.000278)	0.418336 (0.149413)	0.028644
EGARCH (1,2)			
Without REC	0.000082 (0.000297)	0.952780 (0.211536)	0.077116
With REC	-0.000003 (0.000315)	1.005791 (0.221404)	0.084996

The R^2's from these regressions tell us which models best fit the data within sample. For the full period, the TARCH (1,2) model slightly outperforms the EGARCH

(1,2) model with an R^2 of .136 versus .133. The GARCH (1,2) model trails slightly at .121, while the ARCH (2) model fits poorly with an R^2 of .061. The inclusion of the recession dummy tends to improve fit, but only marginally. Results are similar for the middle period, with the TARCH (1,2) performing best (.146), but in this case the GARCH (1,2) fits better than the EGARCH (1,2) (.132 versus .129). The later period is again unusual in that all of the models are much poorer fits. Also, the EGARCH (1,2) model is by far the superior one for this time period with an R^2 of .085 versus .031 for its closest competitor, in this case the ARCH (2) model.

<u>Summary of Stock Return Conditional Heteroskedasticity Evidence</u>

All of this evidence reveals some interesting characteristics of the data. Each sample period exhibits significant persistence in volatility. Also, the effects of positive versus negative shocks on future volatility appears to exhibit significant asymmetry in every case. One of the asymmetric models performs best in every time period, indicating that this asymmetry is a very important characteristic of the data. The persistent uniqueness of the later period also indicates that the behavior of the data has changed over time. Specifically, the post-war data appears to be much less predictable than the earlier data. All of the models presented here perform much more poorly in the most recent period. Evidence on the usually marginal significance of the NBER recession dummy variable indicates that volatility may be slightly higher during recessions, but this does not appear to be a key feature of the data.

<u>Modeling the Conditional Heteroskedasticity of the Risk Premium</u>

Table 6 presents results of estimation of the four models presented above (for the

Table 6

Model coefficient estimates (standard errors in parentheses) for conditional variance of the risk premium for full period.
Estimates are maximum likelihood estimates of the parameters of the appropriate model for the conditional variance.

A. ARCH (2) Model: $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \delta\varepsilon_{t-2}^2 + \rho REC_t$

	ω	α	δ	ρ
Without REC	0.001543 (0.000062)	0.099223 (0.018593)	0.302400 (0.026953)	
With REC	0.001368 (0.000054)	0.078060 (0.016467)	0.248979 (0.024616)	0.001146 (0.000170)

B. GARCH (1,2) Model: $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \delta\varepsilon_{t-2}^2 + \beta\sigma_{t-1}^2 + \rho REC_t$

	ω	α	δ	β	ρ
Without REC	0.000105 (0.000024)	0.035247 (0.025580)	0.103123 (0.031291)	0.823018 (0.019927)	
With REC	0.000092 (0.000024)	0.028172 (0.024188)	0.105394 (0.030720)	0.820140 (0.020320)	0.000111 (0.000039)

C. TARCH (1,2) Model: $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \gamma\varepsilon_{t-1}^2 d_{t-1} + \delta\varepsilon_{t-2}^2 + \beta\sigma_{t-1}^2 + \rho REC_t$

	ω	α	γ	δ	β	ρ
Without REC	1.05E-04 (0.000022)	-0.034953 (0.020852)	0.106124 (0.025405)	0.111829 (0.027614)	0.829290 (0.020358)	
With REC	9.28E-05 (0.000022)	-0.036983 (0.020270)	0.095949 (0.027196)	0.120235 (0.027802)	0.825176 (0.020637)	0.000081 (0.000039)

D. EGARCH (1,2) Model: $\ln\sigma_t^2 = \omega + \alpha\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma\frac{\varepsilon_{t-1}}{\sigma_{t-1}} + \delta\left|\frac{\varepsilon_{t-2}}{\sigma_{t-2}}\right| + \lambda\frac{\varepsilon_{t-2}}{\sigma_{t-2}} + \beta\ln\sigma_{t-1}^2 + \rho REC_t$

	ω	α	γ	δ	λ	β	ρ
Without REC	-0.354392 (0.049561)	0.007830 (0.050613)	-0.203831 (0.035345)	0.219964 (0.056277)	0.139438 (0.029998)	0.971282 (0.006407)	
With REC	-0.397123 (0.050266)	-0.034090 (0.053793)	-0.226466 (0.036448)	0.248986 (0.057831)	0.184086 (0.032317)	0.965546 (0.006536)	0.059495 (0.015635)

full period only) applied to the risk premium (stock returns minus interest rates). Panel A can be compared to Panel A of Table 1, Panel B to Panel A of Table 2, etc. Comparing the results for the risk premium to those for stock returns, there do not appear to be any significant differences. The results follow the same pattern in each case, with similar coefficient estimates and similar significance levels in every case. Table 7 provides analogous regression results to those presented in Table 5 for stock returns (again, using only the full period). The conclusions are identical, with the TARCH model performing the best (R^2 of .131) followed by the EGARCH model (R^2 of .130), the GARCH model

Table 7

Comparison of within-sample predictive power of models for the conditional variance of the risk premium.
OLS estimation results of the following equation (White (1980) heteroskedasticity-consistend standard errors in parentheses):

$$\hat{e}_t^2 = \alpha + \beta\hat{\sigma}_t^2 + \upsilon_t$$

A. Full period (1885:01 - 1998:09)

	α	β	R^2
ARCH (2)			
Without REC	0.000880 (0.000532)	0.667913 (0.247283)	0.052376
With REC	0.000521 (0.000602)	0.823636 (0.282825)	0.059625
GARCH (1,2)			
Without REC	0.000387 (0.000486)	0.846570 (0.241348)	0.113986
With REC	0.000321 (0.000498)	0.876077 (0.247565)	0.115794
TARCH (1,2)			
Without REC	-0.000031 (0.000558)	1.016170 (0.272887)	0.134224
With REC	0.000022 (0.000545)	0.991388 (0.266476)	0.131465
EGARCH (1,2)			
Without REC	-0.000181 (0.000569)	1.100110 (0.283106)	0.129022
With REC	0.000083 (0.000544)	0.978937 (0.266489)	0.129615

(R^2 of .116) and finally the ARCH (2) model (R^2 of .060). These results are not surprising. While the volatility of the risk premium will be different than that of stock returns (by simple statistical operators, Var(Risk Premium)=Var(Stock Returns)+Var(Interest Rates)-2Cov(Stock Returns, Interest Rates)), most of the variability of the risk premium will come from the variability of stock returns. The Interest Rate series is much more stable than the stock return series.

Current Volatility

Figure 1 presents a plot of monthly conditional volatility for the full sample period using results from the TARCH (1,2) model (including the recession dummy). Obviously, as compared to the entire period, volatility in the most recent years is not particularly high. Even looking at the period since 1970, the last several years of the sample actually seem to exhibit fairly low conditional volatility.

Figure 1:
Fitted Conditional Volatility Values