FIN 533

Introduction to ARIMA Models

- Differencing to induce stationarity
- Implications for long-term behavior
- Implications for forecasting

ARIMA(0,1,1):

$$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$$

Random shock form:

$$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$$

$$Z_t = a + a_t - q_1 a_{t-1} + Z_{t-1}$$

$$= a + a_t - q_1 a_{t-1} + (a + a_{t-1} - q_1 a_{t-2} + Z_{t-2})$$

$$= 2a + a_t + (1 - q_1)\, a_{t-1} - q_1 a_{t-2} + Z_{t-2}$$

$$= 2a + a_t + (1 - q_1)\, a_{t-1} - q_1 a_{t-2} + (a + a_{t-2} - q_1 a_{t-3} + Z_{t-3})$$

$$= 3a + a_t + (1 - q_1)\, a_{t-1} + (1 - q_1)\, a_{t-2} + q_1 a_{t-3} + Z_{t-3}$$

. . . . .

ARIMA(0,1,1): $(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$

Random shock form (cont.):

$Z_t = a\ t + a_t + (1-q_1)\,[a_{t-1} + a_{t-2} + \ldots + a_1] - q_1 a_0 + Z_0$

- **which is a time trend, starting at Z_0, adding $(1-q_1)$ times the sum of the accumulated shocks**
  - **like a random walk plus a time trend (if there is drift)**

ARIMA(0,1,1): $(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$

Autoregressive form (ignoring constant):

$(Z_t - Z_{t-1}) = a_t - q_1 a_{t-1}$

$Z_t = a_t + Z_{t-1} - q_1 a_{t-1}$

$= a_t + Z_{t-1} - q_1 (Z_{t-1} - Z_{t-2} + q_1 a_{t-2})$

$= a_t + (1-q_1) Z_{t-1} - q_1 Z_{t-2} + q_1^2 a_{t-2}$

$= a_t + (1-q_1) Z_{t-1} - q_1 Z_{t-2} + q_1^2 (Z_{t-2} - Z_{t-3} + q_1 a_{t-3})$

$= a_t + (1-q_1) Z_{t-1} + q_1 (1-q_1) Z_{t-2} - q_1^2 Z_{t-3} + q_1^3 a_{t-3}$

. . . . .

ARIMA(0,1,1):

$$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$$

Autoregressive form (ignoring constant):

$$Z_t = a_t + (1-q_1) Z_{t-1} + q_1 (1-q_1) Z_{t-2} + q_1^2 (1-q_1) Z_{t-3} +$$

$$q_1^3 (1-q_1) Z_{t-4} + q_1^4 (1-q_1) Z_{t-5} + \ldots$$

- **which is an exponentially weighted average of the past data, declining at the rate** q_1

$$Z_t = a_t + (1-q_1)[Z_{t-1} + q_1 Z_{t-2} + q_1^2 Z_{t-3} + q_1^3 Z_{t-4} + \ldots]$$

ARIMA(0,1,1):

$$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$$

Autoregressive weights:

$$p_k = q_1^{k-1}(1-q_1)$$

- **the autoregressive coefficients follow a geometric series, so the sum of the coefficients is one:**

  sum = $(1-q_1) / (1-q_1) = 1$

In other words, the forecasting function is a proper average of the past data

ARIMA(0,1,1):
$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$

Adaptive forecasting:

where $\hat{Z}_t(1)$ is the one-step-ahead forecast made at time t

$[Z_t - Z_{t-1}] = a_t - q_1 a_{t-1}$

$\hat{Z}_{t-1}(1) = Z_{t-1} - q_1 a_{t-1} = Z_{t-1} - q_1 [Z_{t-1} - \hat{Z}_{t-2}(1)]$

$= (1 - q_1) Z_{t-1} + q_1 \hat{Z}_{t-2}(1)$

ARIMA(0,1,1):
$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$

Adaptive forecasting (cont.):

$\hat{Z}_t(1)$ is a weighted average of the most recent observation and the most recent forecast

- **if q_1 is large, the forecast changes very smoothly**
  - **i.e., the most recent forecast gets most of the weight**
- **if q_1 is small, the forecast changes very quickly**
  - **i.e., the most recent observation gets most of the weight**

ARIMA(0,1,1):

$$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$$

Exponential Smoothing:

- the ARIMA(0,1,1) model generates forecasts that are equivalent to simple "exponential smoothing", where q_1 is the smoothing parameter
  - instead of specifying this forecasting method on an ad hoc basis, however, we let the data tell us if this is the appropriate forecasting method, and what the appropriate smoothing parameter should be

ARIMA(0,1,1):

$$(Z_t - Z_{t-1}) = a + a_t - q_1 a_{t-1}$$

Adaptive forecasting:

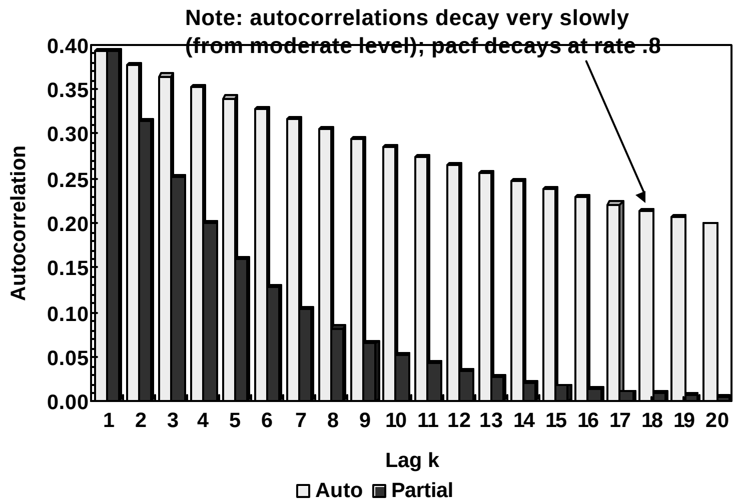
$$\hat{Z}_{t-1}(1) = (1 - q_1)\, Z_{t-1} + q_1\, \hat{Z}_{t-2}(1)$$

$$= (1 - q_1)\, [\, \hat{Z}_{t-2}(1) + a_{t-1}\,] + q_1\, \hat{Z}_{t-2}(1)$$

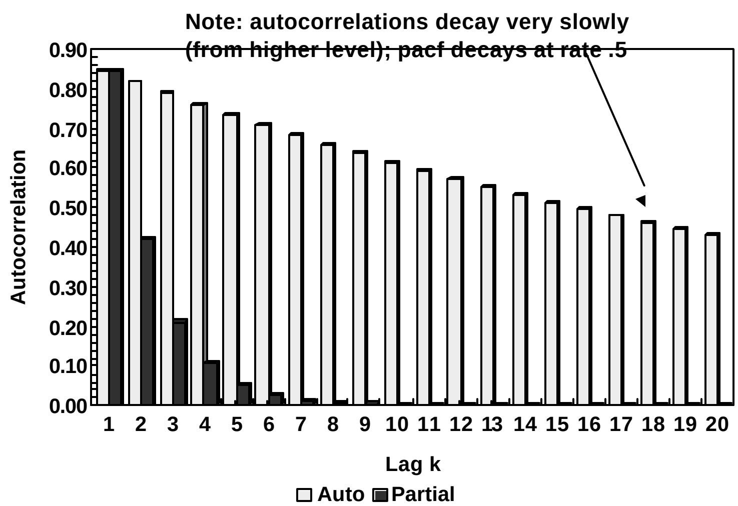
$$= \hat{Z}_{t-2}(1) + (1 - q_1)\, a_{t-1}$$

- so the forecast of future values of an ARIMA(0,1,1) process changes in proportion to the most recent shock or error
  - small changes if q_1 is close to 1

ARIMA(0,1,1): $(Z_t - Z_{t-1}) = a_t - .8\, a_{t-1}$; (T = 150)



ARIMA(0,1,1): $(Z_t - Z_{t-1}) = a_t - .5\, a_{t-1}$; (T = 150)



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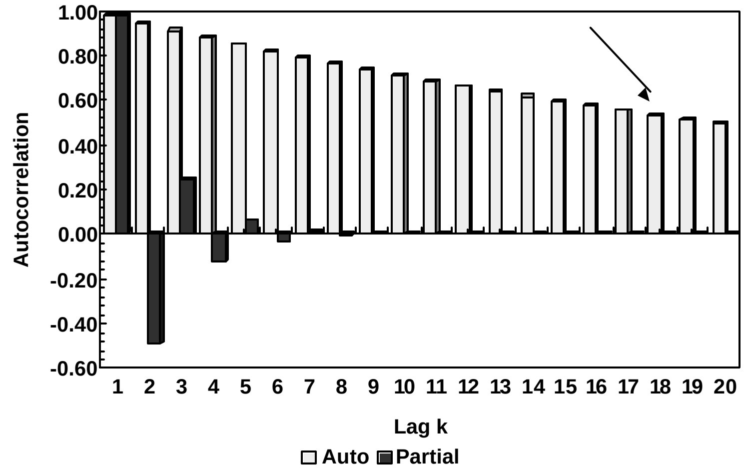
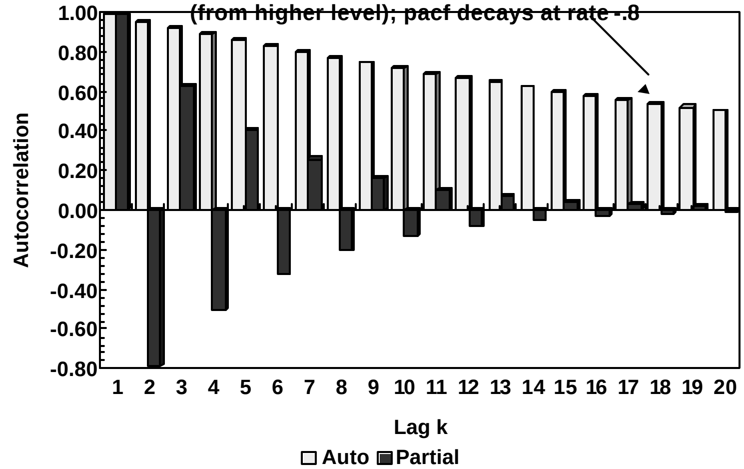


ARIMA(0,1,1): $(Z_t - Z_{t-1}) = a_t + .8\, a_{t-1}$; (T = 150)

Note: autocorrelations decay very slowly (from higher level); pacf decays at rate -.8



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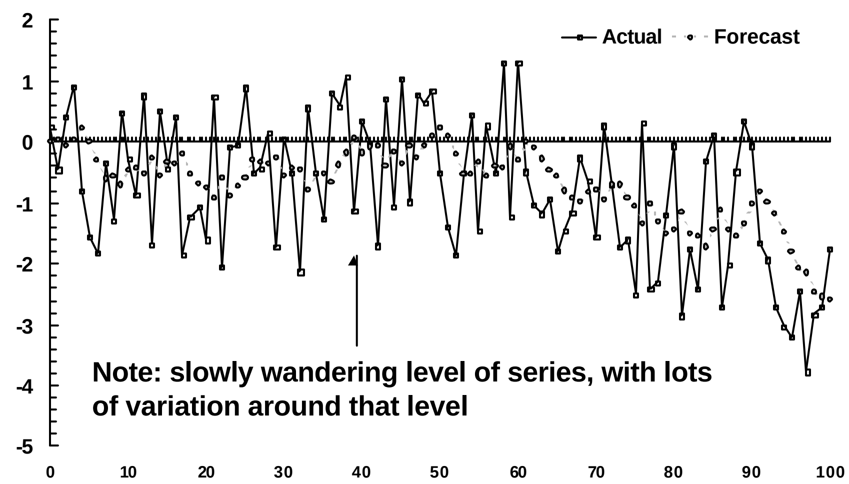

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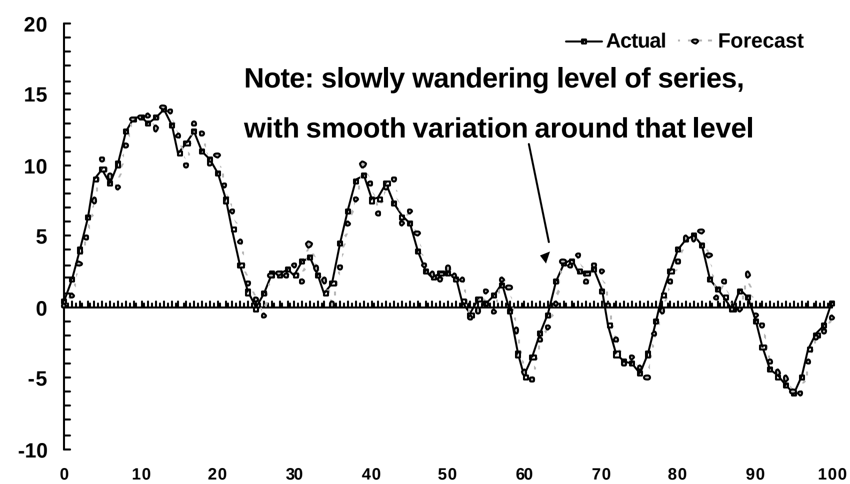

Integrated Moving Average Models: Where Do They Come From?

Random walk plus (independent) noise:

- **=> ARIMA(0,1,1)**
  - **the size of the moving average parameter is determined by the relative size of the variance of the shocks to the random walk versus the variance of the "noise"**
  - **when the noise is small, q is small, and it looks like a random walk**
  - **when the noise is large, q is large, and it looks like a random variable with a slowly wandering level (mean)**

Random Walk Plus Noise: Measurement Error

If the "true" variable follows a random walk, but it is measured with random errors, the "observed" series will follow an ARIMA(0,1,1) model

If expected inflation follows a random walk, actual inflation (expected plus random unexpected inflation) must follow an ARIMA(0,1,1) model

- **this is an implication of the ARIMA(0,1,1) model for CPI inflation**
  - **expected inflation follows a random walk**

Integrated Moving Average Models: Where Do They Come From?

Time aggregated random walk will follow an ARIMA(0,1,1) model

- e.g., if daily stock prices follow a random walk, but S&P reports monthly stock indexes as the average of the daily values of the index, the monthly index will follow an ARIMA(0,1,1)
- you can solve this problem by "point sampling"
  - i.e., measure the prices (at the same time) on the last day of the month (the way CRSP does it)

Integrated Moving Average Models: Where Do They Come From?

Economists often like to think about data as having "components"

- e.g., trend, cycle, seasonal, etc.
- or, "permanent" and "transitory"
  - if "permanent" follows a random walk, and "transitory" is an independent random variable, the observed series is ARIMA(0,1,1)

Integrated Moving Average Models: Summary

1) Autocorrelations decay slowly
   - initial level is determined by how close MA parameter is to one

2) Partial Autocorrelations decay or oscillate
   - determined by MA parameter

3) Always invertible (never stationary)
   - autoregressive weights add to one
   - exponentially weighted moving average of past data

FIN 533

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